Recasting Feynman rules with chirality flow

Thanks to my collaborators:
Joakim Alnefjord, Emil Boman, Andrew Lifson, Christian Reuschle, Simon Plätzer, Adam Warnerbring, Zenny Wettersten

- Inspiration from QCD color
- Dissection of spacetime
- New Feynman rules
- Simplification of calculations, examples
- Conclusion and outlook

Uppsala
October 2024
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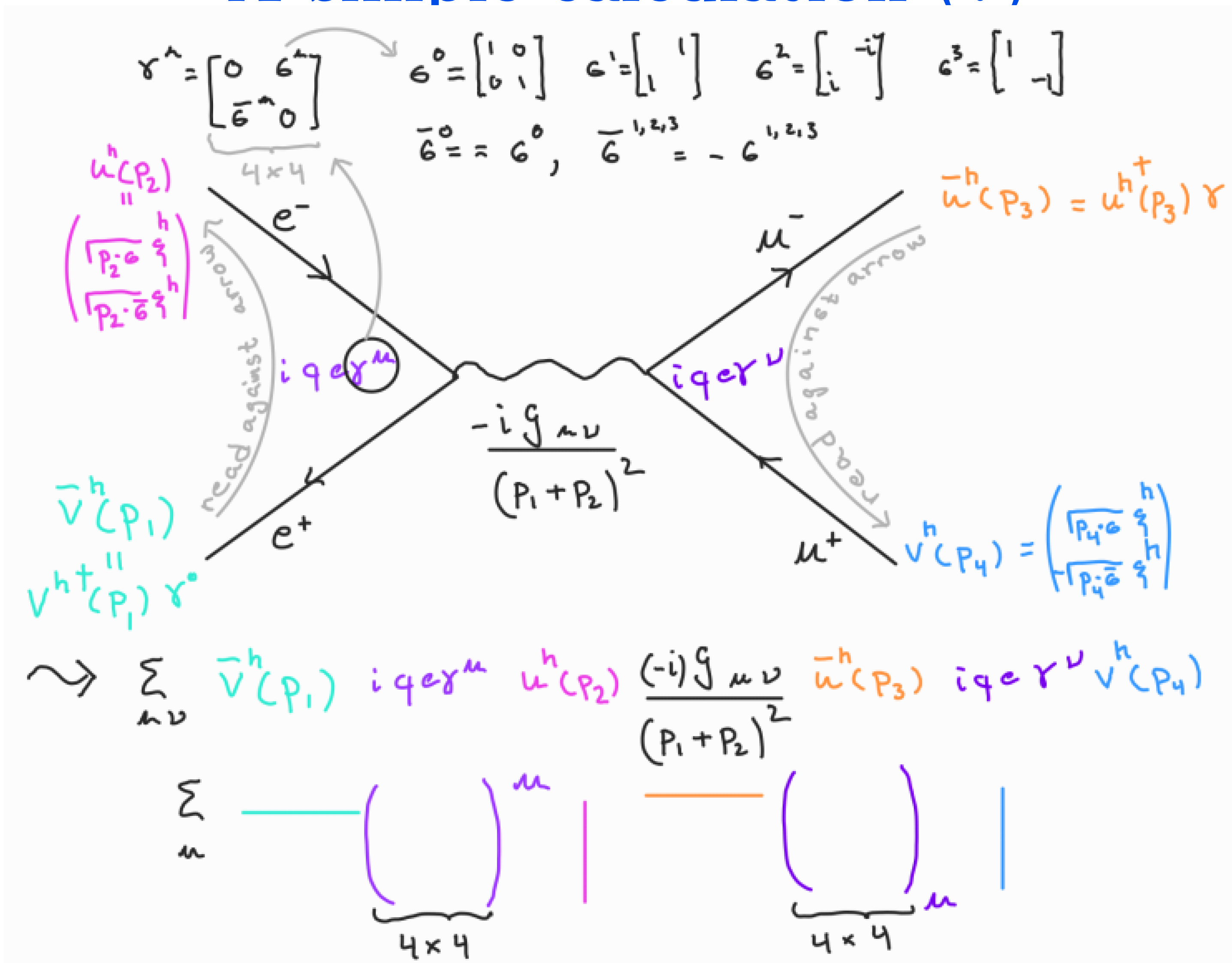
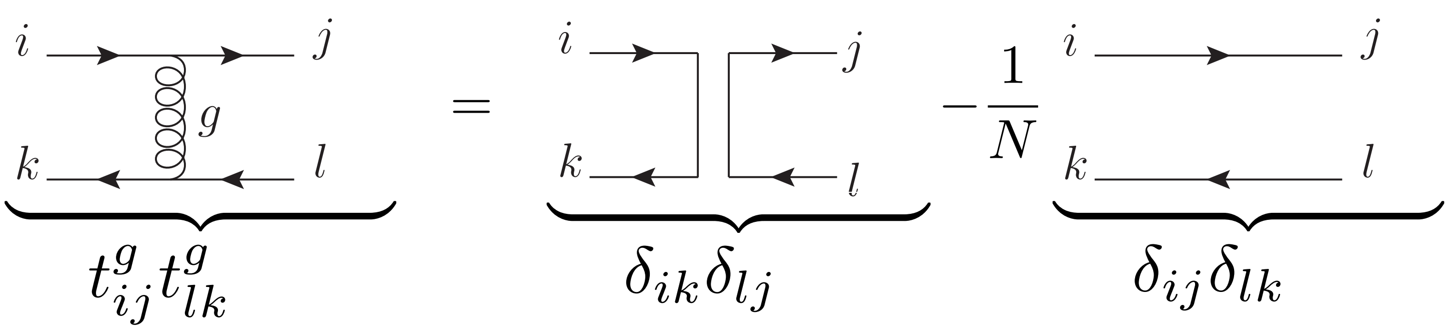
A simple calculation (?)



In QCD we translate color to flows

- For the strong force (QCD) we have color as well: For each quark-gluon vertex a factor t^a_{ij}, a generator of SU(3), i.e., a matrix giving an infinitesimal rotation of a complex three-component vector
- Fierz identity



SU($N=3$) remove gluon indices

- The Dirac spinor structure transforms under the direct sum representation $\underbrace{(\frac{1}{2},0)}_{\text{left}}\oplus\underbrace{(0,\frac{1}{2})}_{\text{right}}$ in the chiral/Weyl basis

$$\begin{pmatrix} u_L \\ u_R \end{pmatrix} \rightarrow \begin{pmatrix} e^{-i\bar{\theta}\cdot\frac{\bar{\sigma}}{2}+\bar{\eta}\cdot\frac{\bar{\sigma}}{2}} & 0 \\ 0 & e^{-i\bar{\theta}\cdot\frac{\bar{\sigma}}{2}-\bar{\eta}\cdot\frac{\bar{\sigma}}{2}} \end{pmatrix} \begin{pmatrix} u_L \\ u_R \end{pmatrix}$$

i.e. actually two copies of **SL(2,$\mathbb{C}$)**, infinitesimally two su(2)

- Consider the matrix

$$p^{\dot{\alpha}\beta} = (p_\mu \sigma^\mu)^{\dot{\alpha}\beta} = \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix}^{\dot{\alpha}\beta}$$

If we transform this as indicated by the indices $\Lambda^{\dot{\alpha}}{}_{\dot{\beta}}\Lambda^{\alpha}{}_{\beta}p^{\dot{\beta}\beta}$, we recover the transformation of a four-vector!

- Lorentz group $\sim$ two copies of su(2), $so(3,1) \cong su(2) \oplus su(2)$
- Can we do something similar for the Lorentz structure?

- Consider massless particles: chirality $\sim$ helicity
- Spinors (in notation from the spinor-helicity formalism)

$$u^+(p) = v^-(p) = \begin{pmatrix} 0 \\ |p\rangle \end{pmatrix} \qquad u^-(p) = v^+(p) = \begin{pmatrix} |p] \\ 0 \end{pmatrix}$$

$$\bar{u}^+(p) = \bar{v}^-(p) = \begin{pmatrix} [p|\,, & 0 \end{pmatrix} \qquad \bar{u}^-(p) = \bar{v}^+(p) = \begin{pmatrix} 0\,, & \langle p| \end{pmatrix}$$

- Amplitudes have to be Lorentz invariant
- Lorentz inner products formed using **the only SL(2,$\mathbb{C}$) invariant object** $\epsilon^{\alpha\beta}$, $\epsilon^{12} = -\epsilon^{21} = \epsilon_{21} = -\epsilon_{12}$

$$\underbrace{\epsilon^{\alpha\beta}|i\rangle_\beta}_{\equiv\langle i|^\alpha}|j\rangle_\alpha = \langle i|^\alpha|j\rangle_\alpha = \langle ij\rangle, \quad \underbrace{\epsilon_{\dot\alpha\dot\beta}|i]^{\dot\beta}}_{\equiv[i|_{\dot\alpha}}|j]^{\dot\alpha} = [i|_{\dot\alpha}|j]^{\dot\alpha} = [ij]\ ,$$

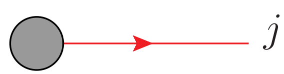
- $\Longrightarrow$ Amplitudes are built up of contractions of form $\langle ij\rangle, [ij] \sim \sqrt{s_{ij}}$
- If we manage to create a flow picture, the "flow" must contract left and right indices separately

- First step: Notation for spinor inner products (only possible Lorentz invariants)

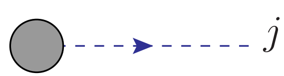
$$\langle i|^{\alpha}|j\rangle_{\alpha} \equiv \langle ij\rangle = -\langle ji\rangle =$$ 

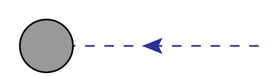
$$[i|_{\dot\beta}|j]^{\dot\beta} \equiv [ij] = -[ji] =$$ 

- Spinors and Kronecker deltas

$$|j\rangle_{\alpha} =$$ 

, $$\langle i|^{\alpha} =$$ 

$$|j]^{\dot\beta} =$$ 

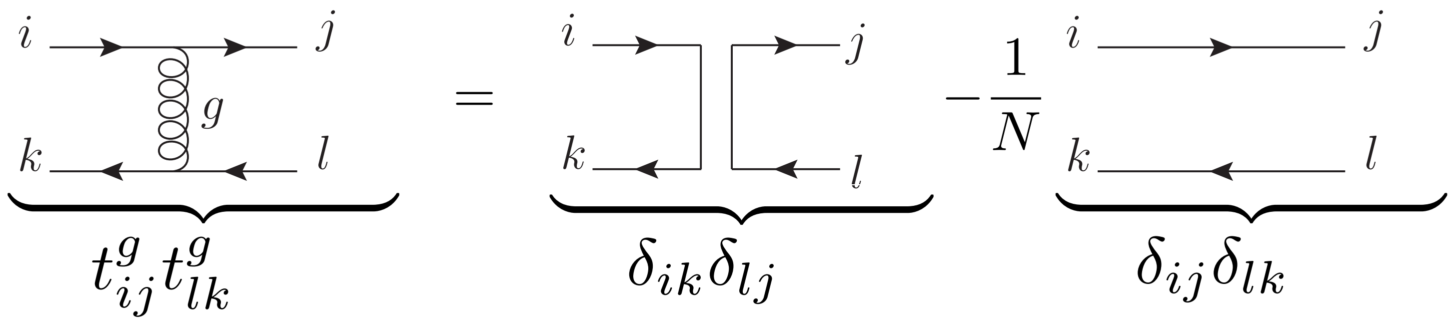
, $$[i|_{\dot\beta} =$$ 

$$\delta_{\alpha}{}^{\beta} \equiv \mathbb{1}_{\alpha}{}^{\beta} =$$ 

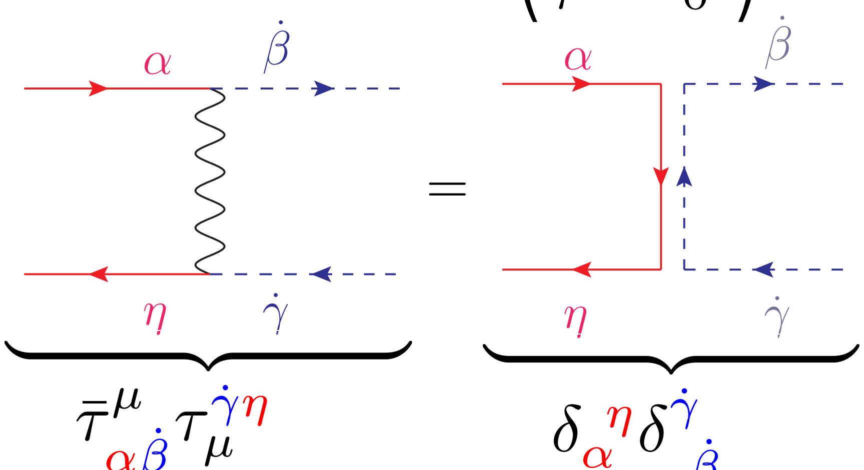
, $$\delta^{\dot\beta}{}_{\dot\alpha} \equiv \mathbb{1}^{\dot\beta}{}_{\dot\alpha} =$$ 

Towards chirality flow: Photon exchange

- Recall color, a single $SU(N)$: generators $t^a \to \delta$'s

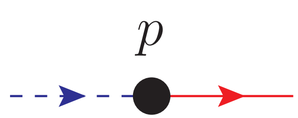


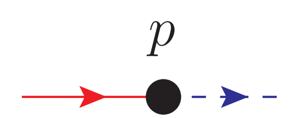
- For the Lorentz structure $\gamma^\mu = \sqrt{2}\begin{pmatrix} 0 & \tau^\mu \\ \bar{\tau}^\mu & 0 \end{pmatrix}$ in vertices



Fermion propagators

- We split $\not{p}_{4d} \equiv p_\mu \gamma^\mu = p_\mu \sqrt{2} \begin{pmatrix} 0 & \tau^\mu \\ \bar{\tau}^\mu & 0 \end{pmatrix}$ into two terms

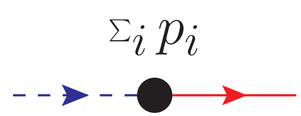
$\not{p} \equiv \sqrt{2} p^\mu \tau_\mu^{\dot{\alpha}\beta} =$ 

$\bar{\not{p}} \equiv \sqrt{2} p_\mu \bar{\tau}^\mu_{\alpha\dot{\beta}} =$ 

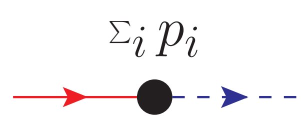
- For massless momenta we have

$$\sqrt{2} p^\mu \tau_\mu \equiv \not{p} = |p]\langle p| \ , \quad \sqrt{2} p^\mu \bar{\tau}_\mu \equiv \bar{\not{p}} = |p\rangle[p|$$

- In a propagator, we have $p^\mu = \sum p_i^\mu \ , \ p_i^2 = 0$

$\not{p} =$ 

$= \sum_i |i]^{\dot{\alpha}} \langle i|^{\beta}$ for $p_i^2 = 0$

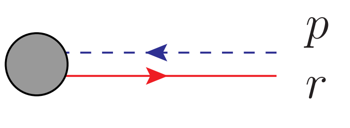
$\bar{\not{p}} =$ 

$= \sum_i |i\rangle_\alpha [i|_{\dot{\beta}}$ for $p_i^2 = 0$

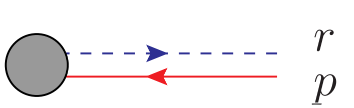
External gauge bosons

- In the spinor-helicity formalism

$$\epsilon_L{}^{\mu}(p,r) \to \frac{|r\rangle[p|}{\langle rp\rangle} = \frac{1}{\langle rp\rangle}$$



$$\epsilon_R{}^{\mu}(p,r) \to \frac{|r]\langle p|}{[pr]} = \frac{1}{[pr]}$$

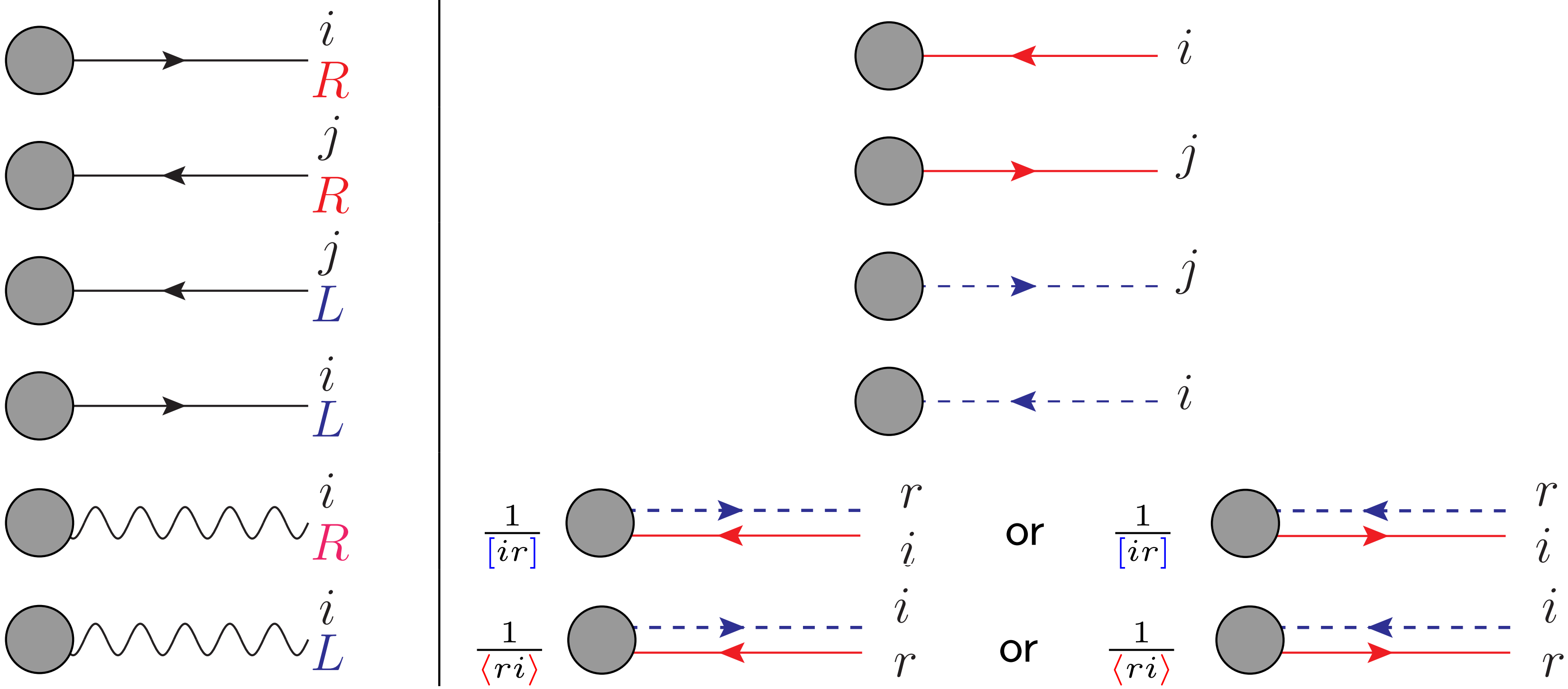


- In a Feynman diagram choose arrow directions which give aligned flows but opposing arrows for gauge bosons

After careful consideration we conclude that this flow picture always works

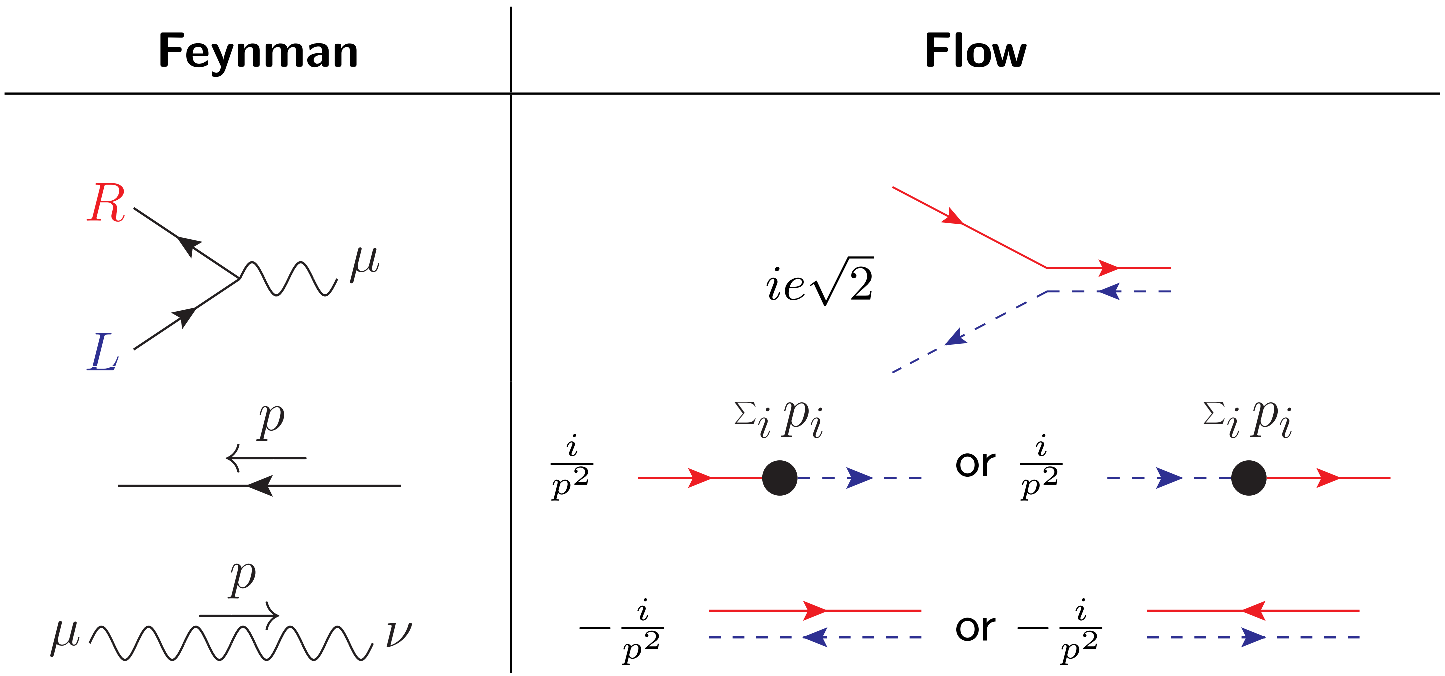
The QED flow rules: massless particles

Species	Feynman	Flow
$\bar{u}^R(p_i)$		
$v^R(p_j)$		
$v^L(p_j)$		
$\bar{u}^L(p_i)$		
${\epsilon_R}^\mu(p_i, r)$		$\frac{1}{[ir]}$ or $\frac{1}{[ir]}$
${\epsilon_L}^\mu(p_i, r)$		$\frac{1}{\langle ri \rangle}$ or $\frac{1}{\langle ri \rangle}$



Use L and R chiral spinors $\underbrace{su(2)}_{\text{dotted}}$, $\underbrace{su(2)}_{\text{undotted}}$ for incoming $-/+$ or outgoing $+/-$ helicity (A. Lifson, C. Reuschle and MS, 2003.05877 (EPJC))

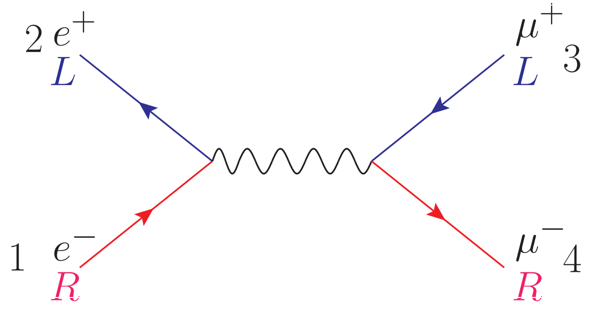
The QED flow rules: vertices and propagators



Stitch together such that arrow directions match

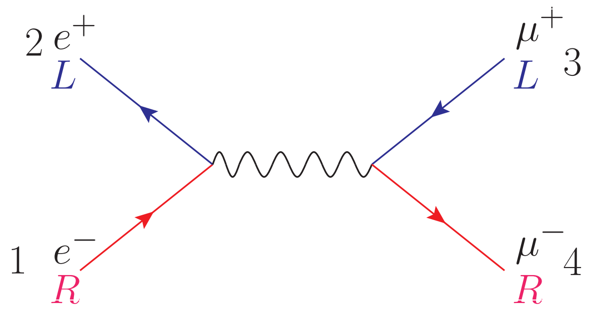
Simplest QED example

- Ordinary Feynman diagrams



$$= \frac{ie^2}{s_{e^+e^-}} \bar{v}^+(p_2)\gamma^\mu u^-(p_1)\bar{u}^+(p_4)\gamma_\mu v^-(p_3) = \ldots$$

- Regular spinor-helicity, easy

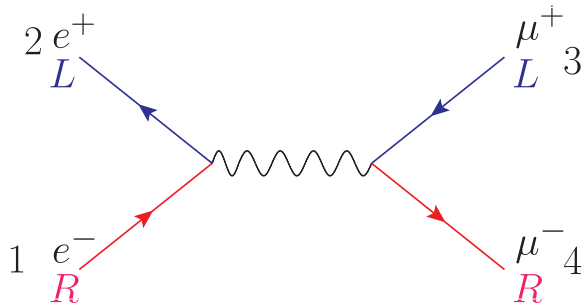


$$= \frac{2ie^2}{s_{e^+e^-}}([2|_{\dot{\alpha}}\tau_\mu^{\dot{\alpha}\beta}|1\rangle_\beta)(\langle 4|^\alpha \bar{\tau}^\mu_{\alpha\dot{\beta}}|3]^{\dot{\beta}})$$

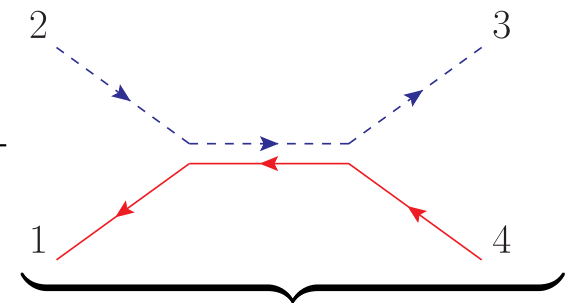
$$= \frac{2ie^2}{s_{e^+e^-}}[2|_{\dot{\alpha}}|3]^{\dot{\alpha}}\langle 4|^\beta|1\rangle_\beta \equiv \frac{2ie^2}{s_{e^+e^-}}[2\,3]\langle 4\,1\rangle$$

Simplest QED example

- Chirality flow, super easy and intuitive

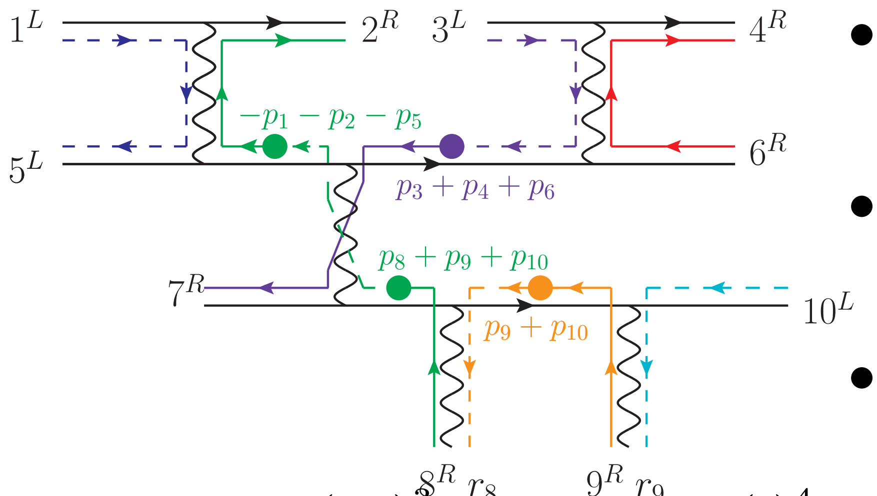


$$= \frac{2ie^2}{s_{e^+e^-}}$$



$$[2\,3]\langle 4\,1\rangle$$

A complicated QED example



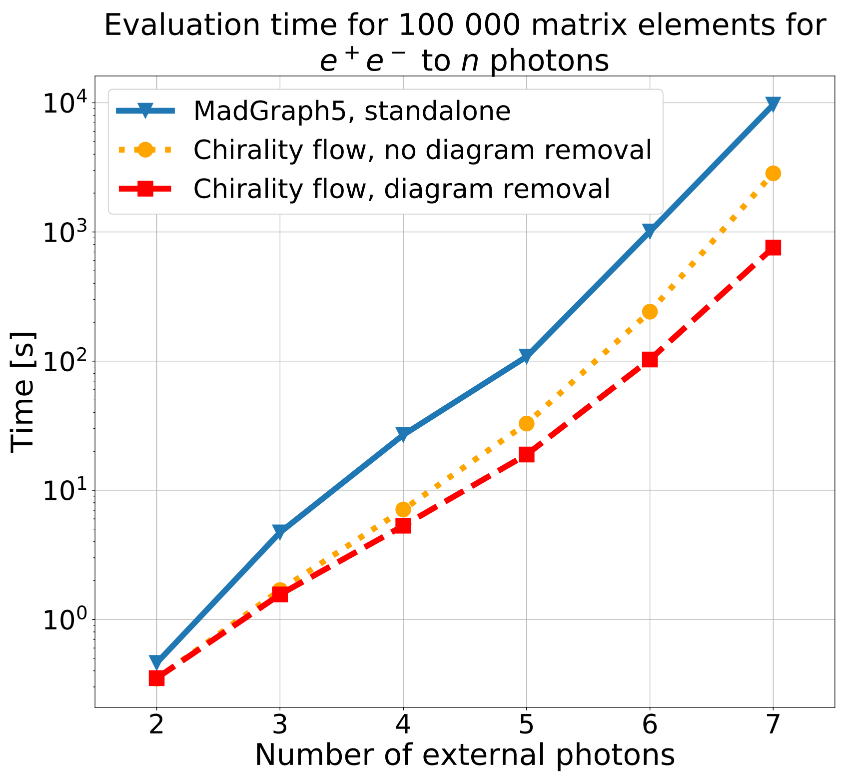
- Pick any consistent arrow direction
- Compare to standard QFT: 12 γ^μ matrices
- Here all particles crossed to outgoing

$$= \underbrace{(\sqrt{2}ei)^8}_{\text{vertices}} \underbrace{\frac{(-i)^3}{s_{1\,2}\ s_{3\,4}\ s_{8\,9\,10}}}_{\text{photon propagators}} \underbrace{\frac{(i)^4}{s_{1\,2\,5}\ s_{3\,4\,6}\ s_{8\,9\,10}\ s_{9\,10}}}_{\text{fermion propagators}} \underbrace{\frac{1}{[8r_8][9r_9]}}_{\text{polarization vectors}} \quad [15]\langle 64\rangle[10\ r_9]$$

$$\times \Big(\underbrace{\langle 9\,9\rangle}_{0}[9r_8] + \langle 910\rangle[10r_8] \Big) \Big(\underbrace{[33]}_{0}\langle 37\rangle + [34]\langle 47\rangle + [36]\langle 67\rangle \Big)$$

$$\times \Big(-\langle 89\rangle[91]\langle 12\rangle - \langle 89\rangle[95]\langle 52\rangle - \langle 8\,10\rangle[10\ 1]\langle 12\rangle - \langle 8\,10\rangle[10\ 5]\langle 52\rangle \Big)$$

Implementation



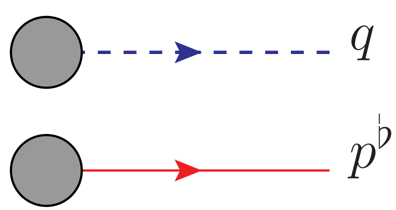
Evaluation time in MadGraph5_aMC@NLO, $e^+e^- \to n$ photons

(A. Lifson, M. Sjodahl and Z. Wettersten 2203.13618 (EPJC))

Massive momenta and spinors

- Decompose massive momentum p as sum of massless
  $p^\mu = p^{\flat,\mu} + \alpha q^\mu \ , \quad (p^\flat)^2 = q^2 = 0 \ , \quad \alpha = \frac{p^2}{2p\cdot q} = \frac{m^2}{2p\cdot q}$
  $\not{p} \equiv \sqrt{2} p^\mu \tau_\mu = |p^\flat]\langle p^\flat| + \alpha |q]\langle q|$
- Massive spinors and polarization vectors written in terms of massless Weyl spinors of momentum $p^\flat, q$,

$u^+(p) = \left(-\frac{m}{[qp^\flat]} \right.$



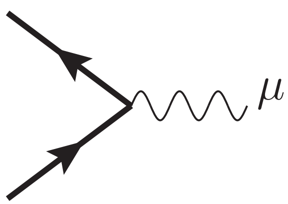
$\left. \right)$, etc.

- q is arbitrary but physical, as it defines the spin direction s^μ

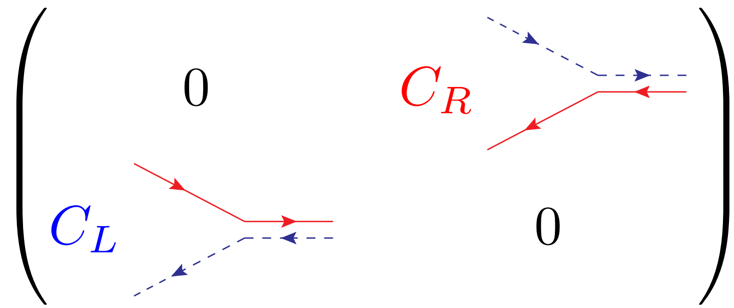
$$s^\mu = \frac{1}{m}(p^\mu - 2\alpha q^\mu) = \frac{1}{m}(p^\mu - \frac{m^2}{p\cdot q} q^\mu)$$

Fermion vertices

- Fermion-vector vertex



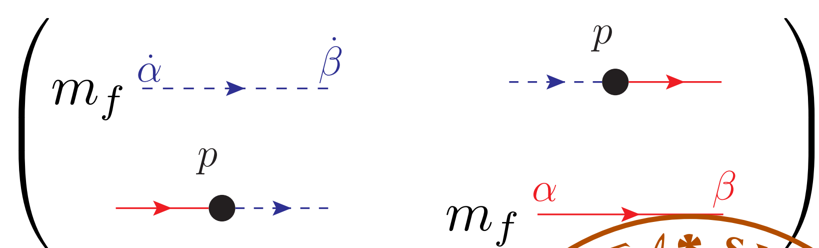
$$= ie(P_L C_L + P_R C_R)\gamma^\mu = ie\sqrt{2}$$



Left and right chiral couplings may differ, in particular $C_R = 0$ for $W^\pm$ $\Rightarrow$ **electroweak sector nice** in chirality flow

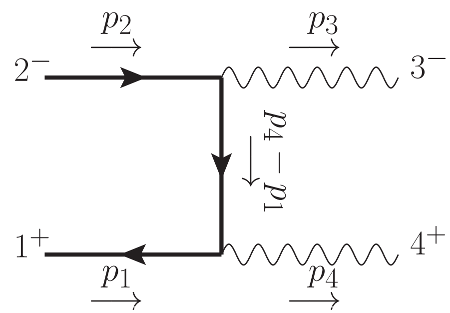
- Fermion propagator

$$\frac{i}{p^2 - m_f^2}\begin{pmatrix} m_f \delta^{\dot{\alpha}}{}_{\dot{\beta}} & \sqrt{2} p^{\dot{\alpha}\beta} \\ \sqrt{2}\bar{p}_{\alpha\dot{\beta}} & m_f \delta_{\alpha}{}^{\beta} \end{pmatrix} = \frac{i}{p^2 - m_f^2}$$



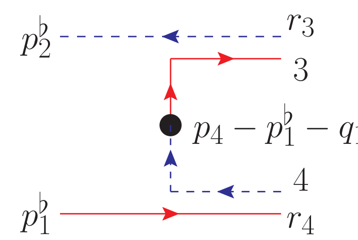
A massive example

Consider the diagram of $e^+_{1+} e^-_{2-} \rightarrow \gamma_3^{-=R} \gamma_4^{+=L}$ and include the mass m_e

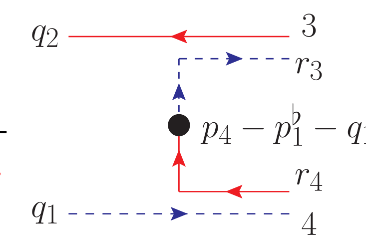


- Obtain 3 new terms
- Simplify with choices of q_1, q_2, r_3, r_4

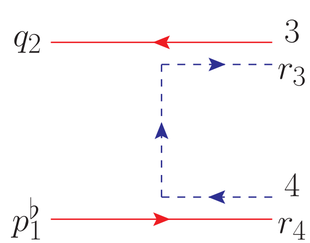
$$= \frac{-2ie^2}{((p_4 - p_1)^2 - m_e^2)[3r_3]\langle 4r_4\rangle} \Bigg\{$$



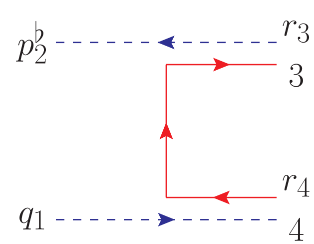
$$- \frac{m_e}{[q_1 p_1^\flat]} \frac{m_e}{\langle p_2^\flat q_2\rangle}$$



$$+ m_e \left(\frac{m_e}{\langle p_2^\flat q_2\rangle}\right.$$

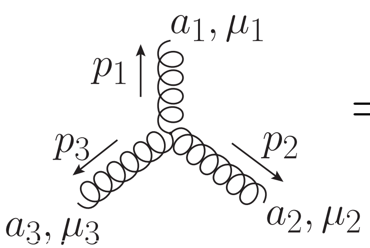


$$- \frac{m_e}{[q_1 p_1^\flat]}$$



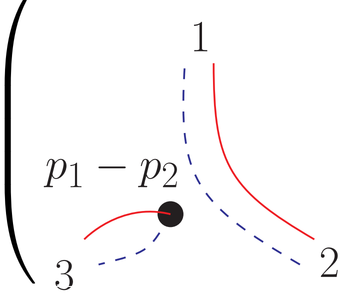
$$\left.\right) \Bigg\}$$

The non-abelian QCD vertices

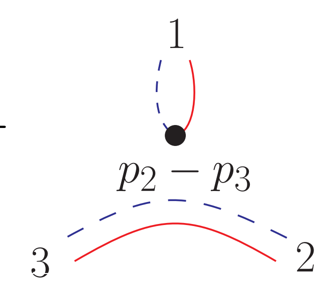


$$= \; i\frac{g_s}{\sqrt{2}} i f^{a_1 a_2 a_3} \left(g^{\mu_1\mu_2}(p_1 - p_2)^{\mu_3} + g^{\mu_2\mu_3}(p_2 - p_3)^{\mu_1} + g^{\mu_3\mu_1}(p_3 - p_1)^{\mu_2} \right)$$

$$\to \; i\frac{g_s}{\sqrt{2}} i f^{a_1 a_2 a_3} \frac{1}{\sqrt{2}} \Bigg($$



$+$



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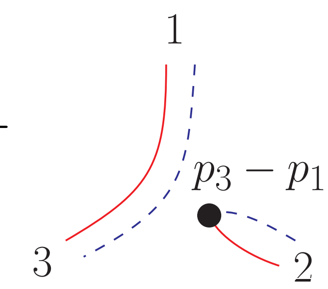
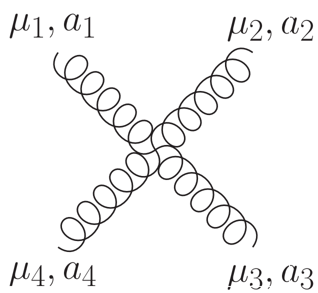


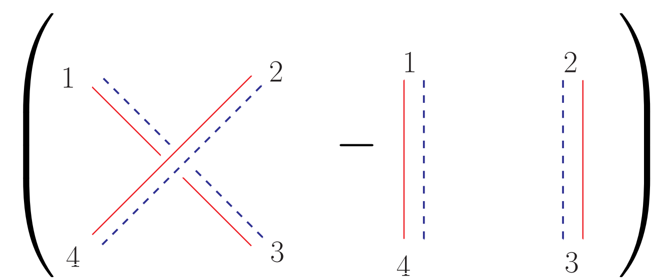
$$\Bigg),$$

Here f^{abc} is the color factor in the vertex, the SU(3) structure constants (A.Lifson, C.Reuschle and M. Sjodahl 2011.10075 (EPJC))

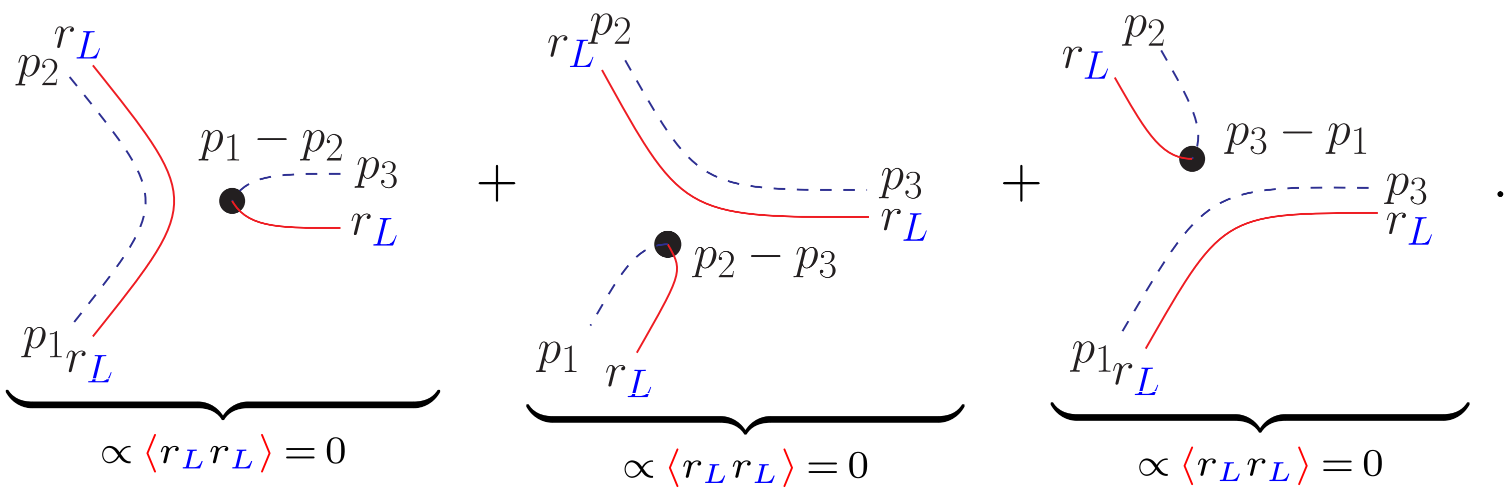


$$= \; i\left(\frac{g_s}{\sqrt{2}}\right)^2 \sum_{Z(2,3,4)} i f^{a_1 a_2 b} i f^{b a_3 a_4} \left(g^{\mu_1\mu_3} g^{\mu_2\mu_4} - g^{\mu_1\mu_4} g^{\mu_2\mu_3}\right)$$

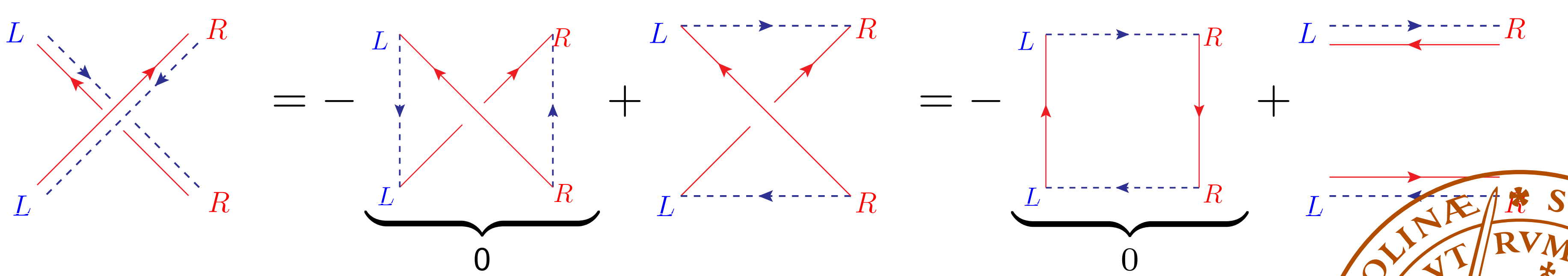
$$\to \; i\left(\frac{g_s}{\sqrt{2}}\right)^2 \sum_{Z(2,3,4)} i f^{a_1 a_2 b} i f^{b a_3 a_4}$$



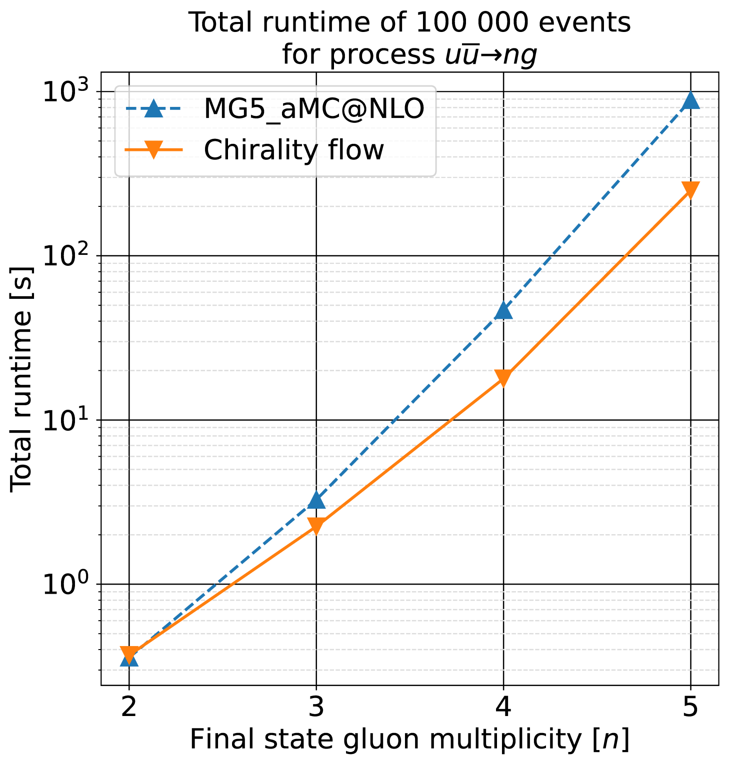
where $Z(2,3,4)$ denotes the set of cyclic permutations of the integers $2, 3$, and 4, The gluon vertices can be simplifed due to a nice gauge choice, for example, we can not have three left gluons



The four-gluon vertex simplifies further due to the Schouten identity

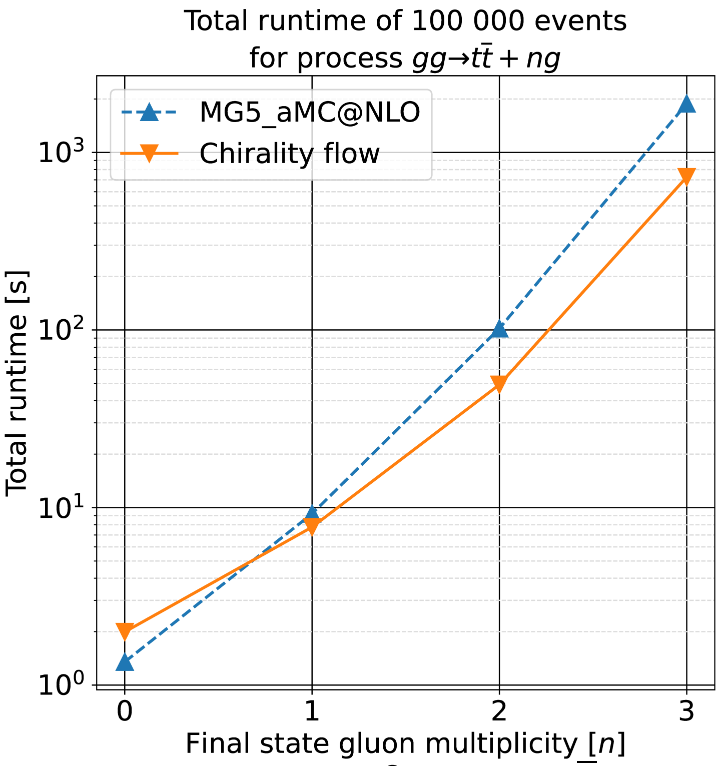


Implementation



Runtime for the Lorentz structure of $u\bar{u} \to ng$ for the standalone version of MadGraph5_aMC@NLO and for the chirality-flow branch. (E. Boman, A. Lifson, M. Sjodahl, A. Warnerbring, Z. Wettersten, 2312.07447 (JHEP))

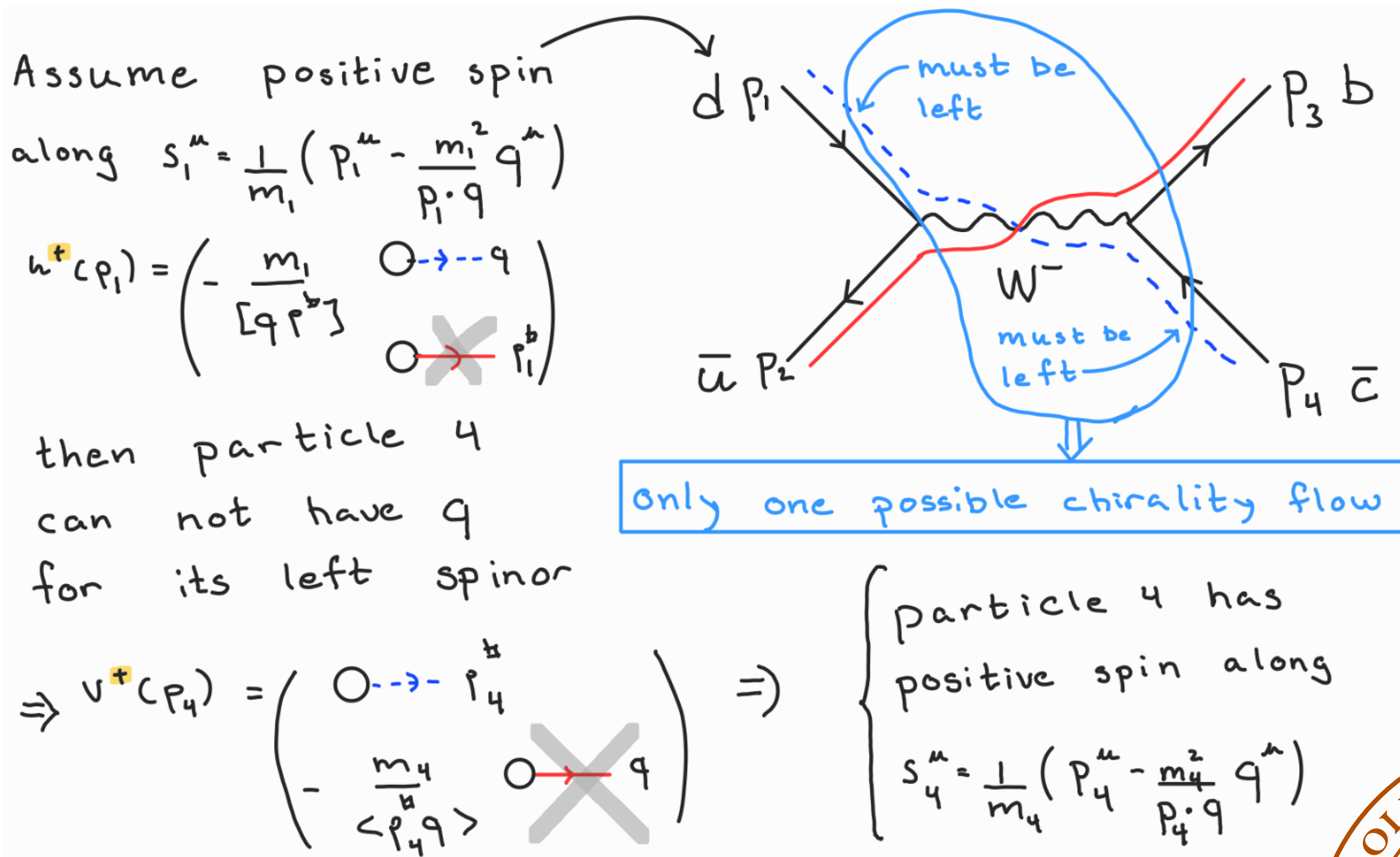
Implementation



Runtime for the Lorentz structure of $gg \to t\bar{t} + ng$, for the standalone version of `MadGraph5_aMC@NLO` and for the chirality-flow branch. (E. Boman, A. Lifson, M. Sjodahl, A. Warnerbring, Z. Wettersten, 2312.07447 (JHEP))

Electroweak physics

- This sector is chiral $\rightarrow$ largest amount of simplification expected
- In particular the $W^{\pm}$ vertex is simple



Conclusion

- Splitting Lorentz structure into su(2), su(2), we are able to recast all standard model Feynman rules to chirality-flow rules
- This gives a transparent and intuitive way of writing down values of Feynman diagrams, simplifying pen-and paper calculations to the extent that they are often trivial
- Significant speed-up for event generation
- Vertices can be simplified by good gauge choices, and the four-gluon vertex further by the Schouten identity

Outlook

- Will explore the (chiral!) electroweak sector
- Further speedups and simplifications in generators?
- Can we formulate the Lagrangian in terms of only spinors?

Backup: What is spin?

- Spin operator Σ^μ and Pauli-Lubanski operator W^μ

$$\frac{1}{2}\Sigma^\mu = -\frac{1}{4m}\epsilon^{\mu\nu\lambda\omega}P_\nu\sigma_{\lambda\omega} = \frac{1}{m}W^\mu$$

- Here $W^2 = -m^2 J(J+1)$, is the 2nd quadratic Casimir operators of the Poincaré algebra, along $P^2 = m^2$, $(P_\nu = i\partial/\partial x^\nu)$, and $\sigma^{\mu\nu}$ (giving the Lorentz generators for the $(1/2, 1/2)$-representation) is defined as $\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]$.
- The spin projected onto s^μ is given by the operator

$$\mathcal{O}_s = -\frac{\Sigma^\mu s_\mu}{2} = \frac{1}{4m}\epsilon^{\mu\nu\lambda\omega}s_\mu P_\nu \sigma_{\lambda\omega} \ ,$$

such that the spin direction, in this sense, is only defined up to a four-momentum proportional to p.

$$\mathcal{O}_s = -\frac{\Sigma^\mu s_\mu}{2} = \frac{1}{4m}\epsilon^{\mu\nu\lambda\omega} s_\mu P_\nu \sigma_{\lambda\omega}$$

- In the non-relativistic limit $p = (m, 0, 0, 0)$ and $s = (0, \hat{s})$. Clearly, adding any linear combination of p to s or q will leave $\mathcal{O}_s$ invariant.
- Adding any linear combination of p to $s = (1/m)(p - 2\alpha q)$ leaves $\mathcal{O}_s$ invariant
- Therefore, we could equally well have used $s' = -2\alpha q/m = -mq/(p \cdot q)$ for the spin direction when expressed as above. (When rewritten as $\mathcal{O}_s = \frac{1}{2}\gamma^5 s^\mu \gamma_\mu$ the spin vector must be taken to be s.)
- Thus q plays the role of defining the *other* four-vector (aside from p) which determines the operator $\mathcal{O}_s$, and thereby what we mean with positive and negative spin.

Backup: Helicity

- Can also measure spin along the direction of motion

$$p^\mu = p_f^\mu + p_b^\mu$$

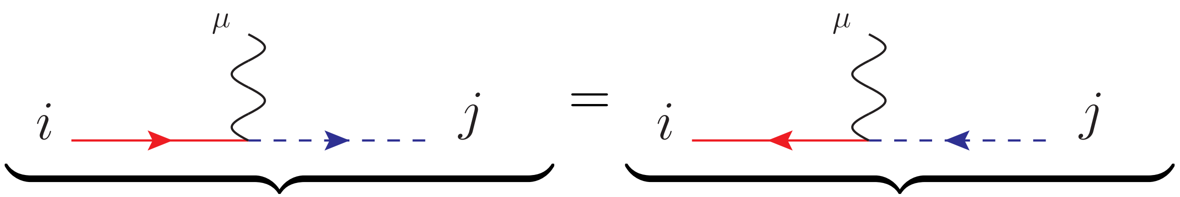
$$\begin{aligned}
\alpha &\rightarrow 1, \\
p^\flat &\rightarrow p_f = \frac{p^0 + |\vec{p}|}{2}(1, \hat{p}) \\
q &\rightarrow p_b = \frac{p^0 - |\vec{p}|}{2}(1, -\hat{p})
\end{aligned}$$

Spin measured along $s^\mu = \frac{1}{m}(p_f^\mu - p_b^\mu) = \frac{1}{m}(|\vec{p}|, p^0\hat{p})$, spatial components along the direction of motion

(Full Standard Model in J. Alnefjord, A.Lifson, C.Reuschle and M. Sjodahl 2011.10075 (EPJC))

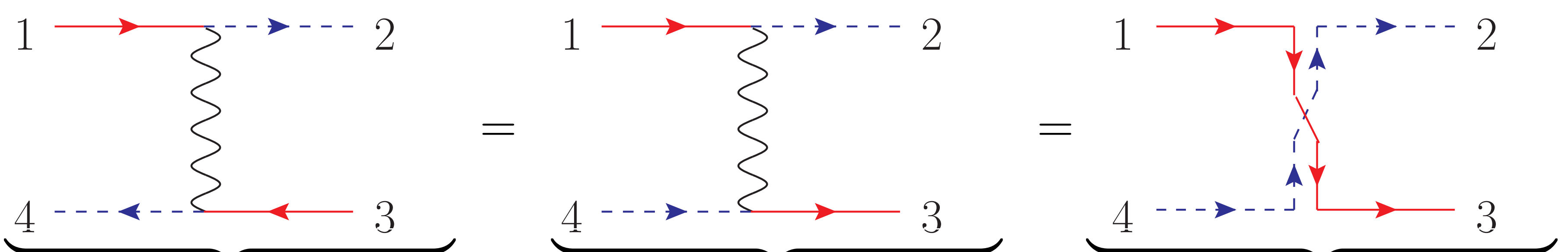
Backup: Non-matching arrows?

Arrows opposed? Flip them (when contracted between external spinors) First use charge conjugation



$\langle i|^{\alpha}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}|j]^{\dot{\beta}}$ $\quad = \quad$ $[j|_{\dot{\alpha}}\tau^{\mu,\dot{\alpha}\beta}|i\rangle_{\beta}$

Then use Fierz identity $\bar{\tau}^{\mu}_{\alpha\dot{\beta}}\tau_{\mu}^{\dot{\gamma}\eta} = \delta_{\alpha}{}^{\eta}\delta^{\dot{\gamma}}{}_{\dot{\beta}}$ to remove vector index

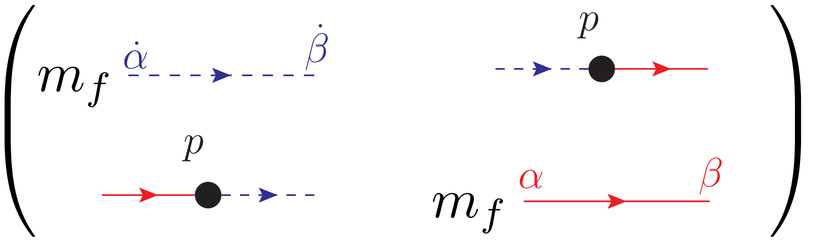


$(\langle 1|^{\alpha}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}|2]^{\dot{\beta}})(\langle 3|^{\gamma}\bar{\tau}_{\mu,\gamma\dot{\eta}}|4]^{\dot{\eta}})$ $\quad = \quad$ $(\langle 1|^{\alpha}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}|2]^{\dot{\beta}})\underbrace{\left(\langle 4|_{\dot{\eta}}\tau_{\mu}^{\dot{\eta}\gamma}|3]_{\gamma}\right)}_{\text{flipped}}$ $\quad = \quad$ $\langle 13\rangle[42]$

Backup: Massive Fermions

- Fermion propagator

$$\frac{i}{p^2 - m_f^2}\begin{pmatrix} m_f \delta^{\dot{\alpha}}{}_{\dot{\beta}} & \sqrt{2} p^{\dot{\alpha}\beta} \\ \sqrt{2}\bar{p}_{\alpha\dot{\beta}} & m_f \delta_{\alpha}{}^{\beta} \end{pmatrix} = \frac{i}{p^2 - m_f^2}$$



- Propagators and vertices don't always contribute factor $\tau/\bar{\tau}$
  $\Rightarrow$ may have even number of $\tau/\bar{\tau}$-matrices
- Have to update arrow swap procedure to include even number of $\tau/\bar{\tau}$. Arrow flips may induce minus signs! Care must be taken

$$\langle i | \bar{\tau}^{\mu_1} \tau^{\mu_2} \ldots \bar{\tau}^{\mu_{2n+1}} | j] = [j | \tau^{\mu_{2n+1}} \bar{\tau}^{\mu_{2n}} \ldots \tau^{\mu_1} | i \rangle$$

$$\langle i | \bar{\tau}^{\mu_1} \tau^{\mu_2} \ldots \tau^{\mu_{2n}} | j \rangle = -\langle j | \bar{\tau}^{\mu_{2n}} \bar{\tau}^{\mu_{2n-1}} \ldots \tau^{\mu_1} | i \rangle$$

$$[i | \tau^{\mu_1} \bar{\tau}^{\mu_2} \ldots \bar{\tau}^{\mu_{2n}} | j] = -[j | \tau^{\mu_{2n}} \bar{\tau}^{\mu_{2n-1}} \ldots \bar{\tau}^{\mu_1} | i]$$

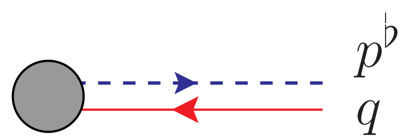
Backup: Massive polarization vectors

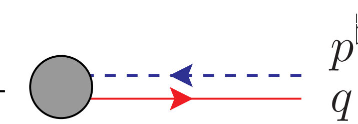
External gauge bosons, for helicity states $p^\flat \to p_f$, $\alpha \to 1$, $q \to p_b$

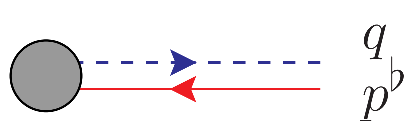
$$\epsilon_L{}^\mu(p) \to \frac{|p^\flat]\langle q|}{\langle q p^\flat\rangle} \quad \text{or} \quad \frac{|q\rangle[p^\flat|}{\langle q p^\flat\rangle} \qquad \epsilon_R{}^\mu(p) \to \frac{|q]\langle p^\flat|}{[p^\flat q]} \quad \text{or} \quad \frac{|p^\flat\rangle[q|}{[p^\flat q]}$$

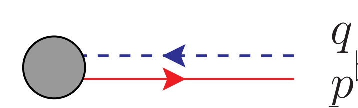
$$\epsilon_0^\mu(p) = s^\mu = \frac{1}{m}(p^{\flat,\mu} - \alpha q^\mu)$$

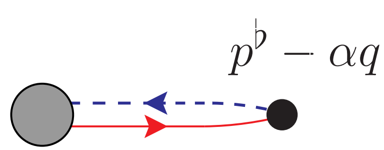
for incoming (outgoing) bosons, use L/R for $-/+$ $(-/+)$ spin along s^μ-axis. Translate to chirality flow

$\epsilon_L{}^\mu(p) \to \frac{1}{\langle q p^\flat\rangle}$ 

, or $\epsilon_L{}^\mu(p) \to \frac{1}{\langle q p^\flat\rangle}$ 

$\epsilon_R{}^\mu(p) \to \frac{1}{[p^\flat q]}$ 

, or $\epsilon_R{}^\mu(p) \to \frac{1}{[p^\flat q]}$ 

$\epsilon_0^\mu(p) \to \frac{1}{m\sqrt{2}}$ 

, or $\epsilon_0^\mu(p) \to \frac{1}{m\sqrt{2}}$