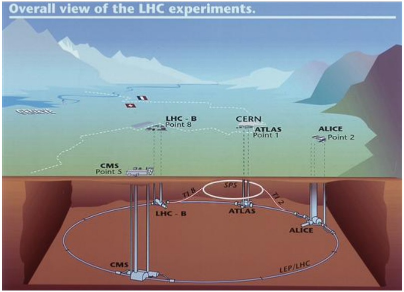
Landau Singularities from Whitney Stratifications

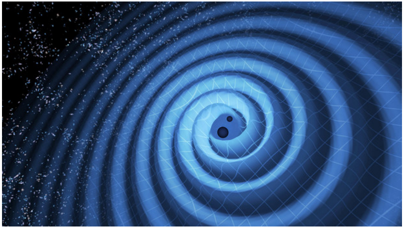
FELIX TELLANDER

Mathematical Institute
University of Oxford

Partikeldagarna, 22 October 2024

Feynman integrals: one integral but much physics





Many other: shadow integrals for confromal blocks, Witten diagrams, cosmological correlators...

Cross section

Collinear factorization:

$$\sigma_{h_1 h_2 \to n} = \sum_{a,b} \int_0^1 dx_a dx_b \, f_{a/h_1}(x_a, \mu_F) f_{b/h_2}(x_b, \mu_F) \hat{\sigma}_{h_1 h_2 \to n}(\mu_F, \mu_R)$$

$$+\mathcal{O}(\Lambda_{\text{QCD}}/Q)$$

Cross section

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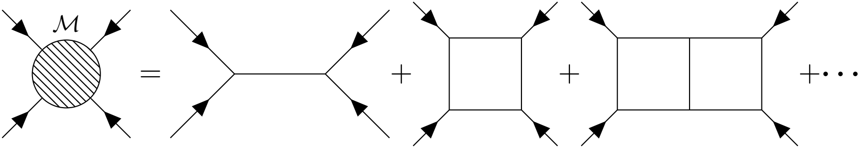
$\mathcal{M}$ is the matrix element and is calculated as a sum of Feynman diagrams:



Feynman integrals

Every (scalar) Feynman integral is an integral of a "rational" function:

$$\mathcal{I}(D, \nu_1, \ldots, \nu_n;\, z) = \int_{\mathbb{R}^n_+} \frac{x_1^{\nu_1} \cdots x_n^{\nu_n}}{\mathcal{G}(z,x)^{D/2}} \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$$

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Questions?

- Mathematical structure - function class, PDE, associated polytopes
- Singularity structure - Landau singularities, symbol alphabets
- Efficient evaluation - closed form and numerical

Singularities of integrals

Singularities are a core ingredient in deriving the CDE [Henn 2013]:

- singularities $\Rightarrow$ symbol alphabet $\Rightarrow$ bootstrap PDE $\Rightarrow$ solve PDE

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Idea to derive singularities:

- Integrals: $\{\text{kinematic variables}\} \times \{\text{integration variables}\} \to \{\text{kinematic variables}\}$
- Topological changes of integrand $+$ map $=$ **singularities**

Stratifying Singular Varieties

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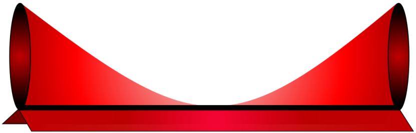
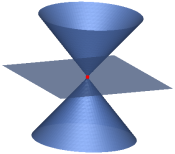
Objects of interest: singular spaces defined by polynomials, such as the figures below.





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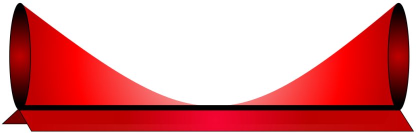

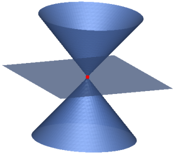

We seek to stratify these spaces by separating them into smooth manifolds which join in a nice way.

Stratifying Singular Varieties

More precisely, for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we will consider algebraic varieties

$$X = \mathbf{V}(I_X) = \mathbf{V}(f_1, \ldots, f_r) = \{p \in \mathbb{K}^n \mid f_1(p) = \cdots = f_r(p) = 0\}.$$

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When $\mathbb{K} = \mathbb{C}$ a point $p \in X$ is singular if the Jacobian matrix of the f_i drops rank at p.

A *stratification* is a filtration, $X_\bullet$, $\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_d = X$ of X s.t. $X = \cup_i X_i$ and s.t. each $\mathcal{M}_i = X_i - X_{i-1}$ is either empty or smooth, i.e. is a manifold, and has pure dimension.

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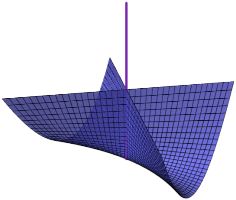
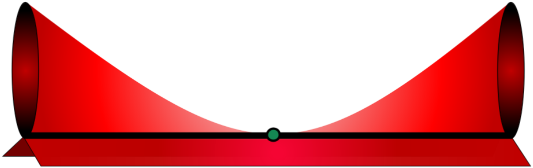
We call the connected components of $\mathcal{M}_i$ *strata*.

Additionally: want decomposition $X = \sqcup_i \mathcal{M}_i$ to be *equisingular*; i.e. the neighbourhood in X of any 2 points of a strata are "similar".



Whitney Stratification

For $X_\bullet$ to be a **Whitney Stratification** these strata must satisfy **Condition B**: for each pair of strata $M, N \subset X$ and a point $y \in N$



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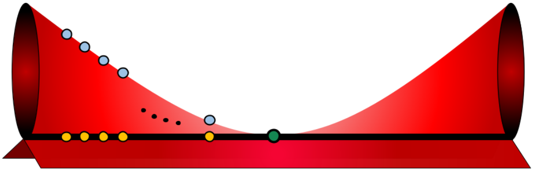

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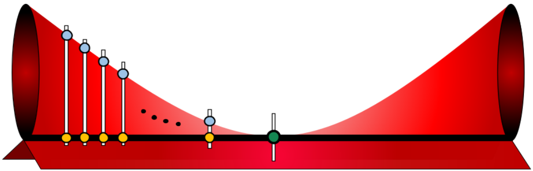

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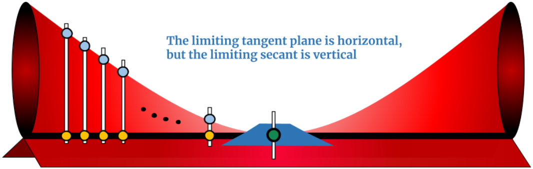

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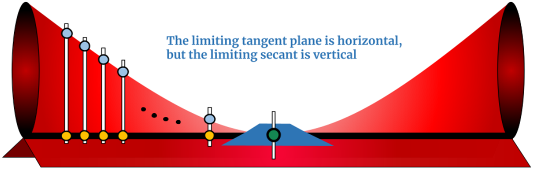

Theorem (H. Whitney, Annals of Math.,1965)

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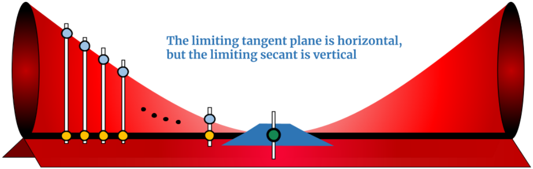

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Goal: given equations defining X **efficiently compute** a Whitney stratification (compute = find equations for each X_i).

Example: Algorithm Applied to the Whitney Umbrella

Several algorithms to compute Whitney stratification have been proposed in the past [Mostowski & Rannou 1991, Rannou 1998, Đinh & Jelonek 2021)]; previous methods have proved impractical on even the smallest examples.
http://martin-helmer.com/Software/WhitStrat/index.html

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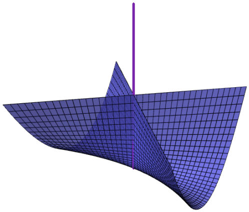

```
needsPackage"WhitneyStratifications"
R = ZZ/10007[x,y,z]
I = ideal(x^2*z-y^2)
time ws = whitneyStratify I
```

```
o5 = MutableHashTable{0 => {ideal (z, y, x)}}
                      1 => {ideal (y, x)}
                                    2     2
                      2 => {ideal(x z - y )}
```

Time: 0.09 seconds, new unpublished improvements, 0.03 seconds

Stratifying Maps

Question: we can stratify real and complex varieties, how about maps?

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Map Stratification:

Let X, Y be algebraic varieties and $f : X \to Y$ an algebraic map. A **stratification of** f, is a Whitney stratification of X and Y so that for every strata S of X there is a strata R of Y such that $f(S) \subset R$ and the derivative, $d(f|_S)$, is surjective where $f|_S$ is the restriction $f|_S : S \to R$.

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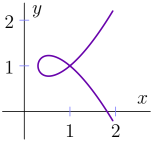
Consequence*: If f is proper: for q, q' in the same stratum N of Y the fibers $f^{-1}(q)$ and $f^{-1}(q')$ have the same topology (i.e. stratified homeomorphism type). This can also be extended to certain dominant maps $f : X \to Y$ in a reasonable way.

Detecting Topological Change in Fibers

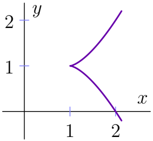
Suppose we wish to study the changes in topology of the curve in $\mathbb{R}^2$ defined by the parametric polynomial

$$f_z(x, y) = (y-1)^2 - (x-z)(x-1)^2$$

in variables x, y with parameter z.



$z < 1$



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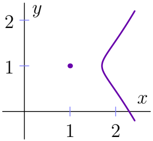

$z > 1$

Detecting Topological Change in Fibers

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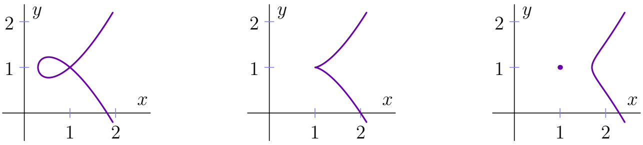
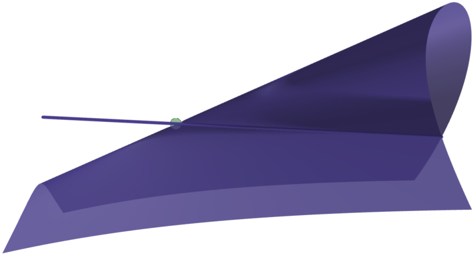


$z < 1$ $\quad$ $z = 1$ $\quad$ $z > 1$

Take $X = \mathbf{V}(f) \subset \mathbb{R}^3$. It is equivalent to ask when the fibers of the projection map $\pi : X \to \mathbb{R}_z$ change topology.

Detecting Topological Change in Fibers

In this case the stratification of the map $\pi : X \to \mathbb{R}_z$ is given by computing a Whitney stratification of X in $\mathbb{R}^3$.



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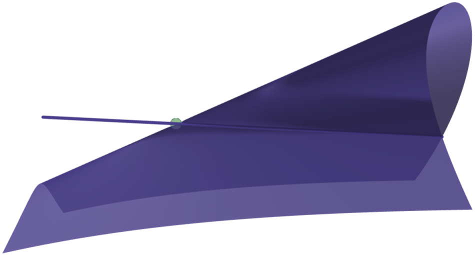

It's Whitney stratification is

$$\{(1, 1, \mathbf{1})\} \subset \mathbf{V}(x - 1, y - 1) \subset X.$$

Hence the topology of the curve changes at $\mathbf{z = 1}$.

Singularities of Feynman Integrals

The **Landau singularities** of a Feynman integral is a variety in the space of parameters z for which the solution to the Feynman integral fails to be an analytic function.

$$\int_{\mathbb{R}_+^{|E|}} \frac{1}{\mathcal{G}^{D/2}} \left(\prod_{e \in E} \frac{x_e^{\nu_e - 1}}{\Gamma(\nu_e)} dx_e \right)$$

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For $\mathcal{G}_h := \mathcal{U}x_0 + \mathcal{F}$ we define:

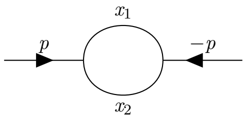
Definition (Helmer, Papathanasiou, FT, 2024)

Set $X = \mathbf{V}(x_0 \cdots x_E \mathcal{G}_h) \subset \mathbb{P}_x^{|E|} \times \mathbb{C}_z^m$, set $Y := \mathbb{C}_z^m$ and consider the projection map $\pi : X \to Y$.
The **Landau variety** is the variety Y_{m-1} appearing in the unique minimal Whitney stratification $(X_\bullet, Y_\bullet)$ of the map π.

Note*: We are compactifying the integration domain by moving from $\mathbb{R}_+^{|E|}$ to the projective simplex $\mathbb{P}_+^{|E|} := \{[x_0 : \cdots : x_{|E|}] \in \mathbb{P}^{|E|} \mid x_e \geq 0 \ \forall \ e = 0, 1, \ldots, |E|\}$.

Example: one-loop bubble

For the bubble graph we have:
$\mathcal{G}_h = x_0(x_1 + x_2) + (m_1^2 + m_2^2 - p^2)x_1x_2 + m_1^2x_1^2 + m_2^2x_2^2$.
Take $X = \mathbf{V}(x_0x_1x_2\mathcal{G}_h) \subset \mathbb{P}^2 \times \mathbb{C}^3$ and let $Y = \mathbb{C}^3$ be the space of kinematic parameters.



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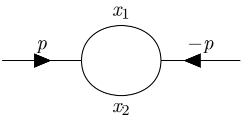


The (minimal) Whitney stratification $(X_\bullet, Y_\bullet)$ of the corresponding projection map $\pi : X \to \mathbb{C}^3$ gives $Y_3 = Y = \mathbb{C}^3$ and,

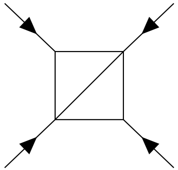
$$Y_2 = \mathbf{V}(m_1^2) \cup \mathbf{V}(m_2^2) \cup \mathbf{V}(p^2) \cup \mathbf{V}(p^4 + m_1^4 + m_2^4 - 2p^2 m_1^2 - 2p^2 m_2^2 - 2m_1^2 m_2^2),$$

$$Y_1 = \mathbf{V}(p^2, m_1^2 - m_2^2) \cup \mathbf{V}(m_2^2 - p^2, m_1^2) \cup \mathbf{V}(m_2^2, m_1^2 - p^2) \cup \mathbf{V}(p^2, m_1^2) \cup \mathbf{V}(p^2, m_2^2) \cup \mathbf{V}(m_2^2, m_1^2),$$

$$Y_0 = \mathbf{V}(p^2, m_1^2, m_2^2).$$

Y_2 is the **Landau variety**.

Example: two-loop slashed box



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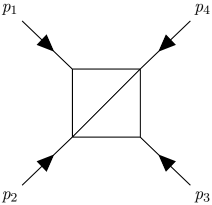

All $m_e = 0$, $p_1^2 = p_2^2 = 0$ and p_3^2, $p_4^2 \neq 0$. The Landau variety consists of 9 components:

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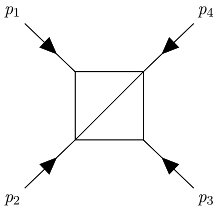

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The component $p_4^2 - s - t$ is missed by the recently proposed *principal Landau determinant*.

Summary

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