New Insights on Non-Perturbative BPS Black Hole Entropy

Alberto Castellano Mora

Strings and Geometry 2026
Uppsala University, May 19th 2026

Based on:

- *Black Hole Entropy, Quantum Corrections and EFT Transitions*,
  A. Castellano, M. Zatti, [arXiV:2502.02655]
- *Quantum Calabi-Yau Black Holes and Non-Perturbative D0-brane Effects*,
  A. Castellano, D. Lüst, C. Montella, M. Zatti,
  [arXiV:2505.15920]
- *On Supersymmetric D-brane probes in 4d N=2 $AdS_2 \times S^2$ Attractors*,
  A. Castellano, C. Montella, M. Zatti, [arXiV:2507.17857]
- *Exact Path Integral Methods in Supersymmetric $AdS_2 \times S^2$ backgrounds*,
  A. Castellano, C. Montella, M. Zatti, [arXiV:2603.15730]
- *How to (Non-)Perturb a BPS Black Hole*,
  A. Castellano, M. Zatti, [arXiV:2604.00116]
- *Work To Appear*, with J. Chu and J. Law-Smith

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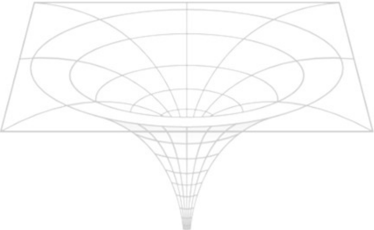
[See also Matteo's poster
this afternoon!]

Why Should we Care about Black Holes?

- (One of the) most ambitious goal(s) of theoretical physics: Understand Quantum Gravity

- Relevant for phenomenology (e.g., hierarchies, cosmology, etc.)!

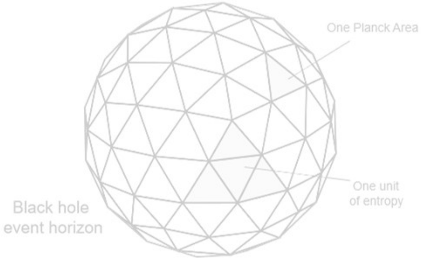
- The 'Hydrogen Atom' of QG: Black Holes



$$ds^2 = -\left(1-\frac{r_h}{r}\right)dt^2 + \left(1-\frac{r_h}{r}\right)^{-1}dr^2 + r^2 d\Omega_2^2$$

- More importantly, their entropy tell us about microscopic DOFs

$$S_{\text{BH}} = \frac{c^3 k_B A_h}{4 G_N \hbar}$$



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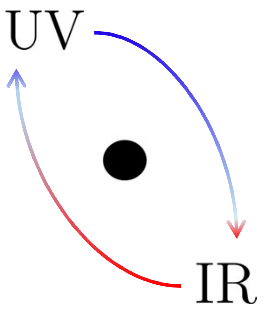

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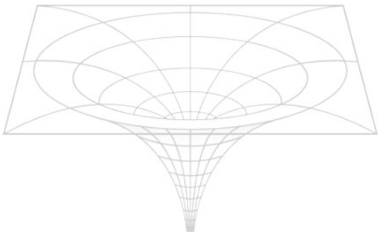

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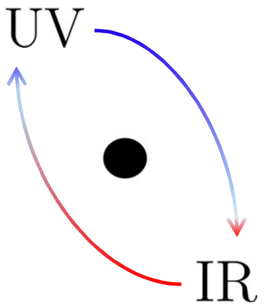

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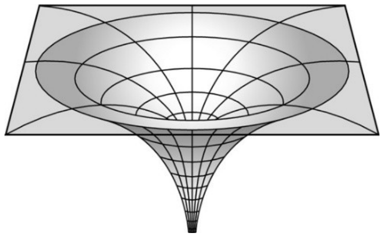

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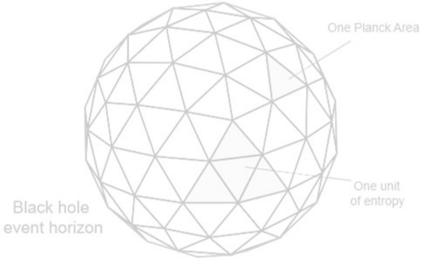

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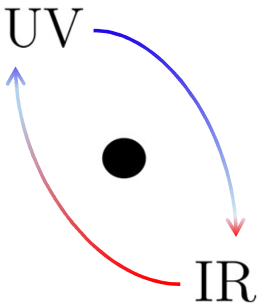

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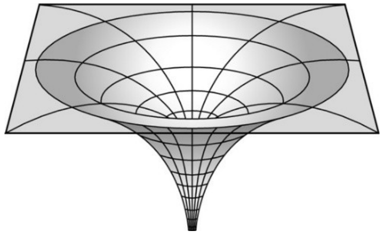

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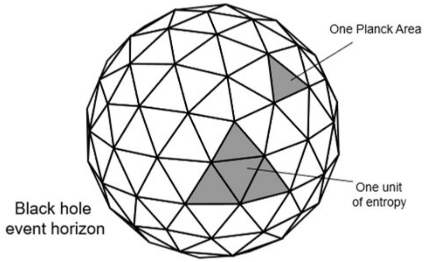

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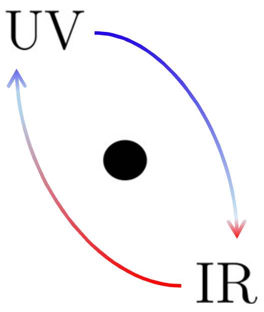

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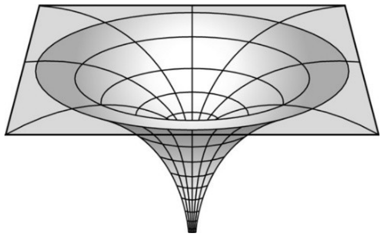

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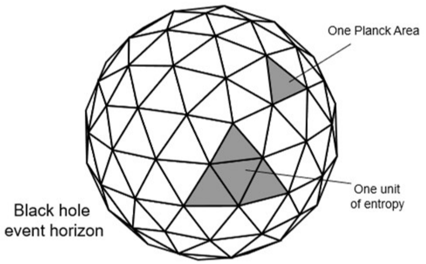
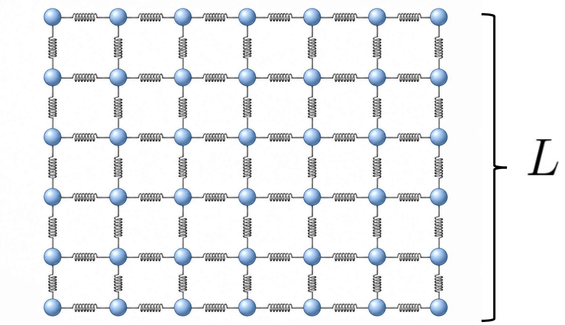
$\rightsquigarrow$ Quantum!



The Two-Derivative Veil



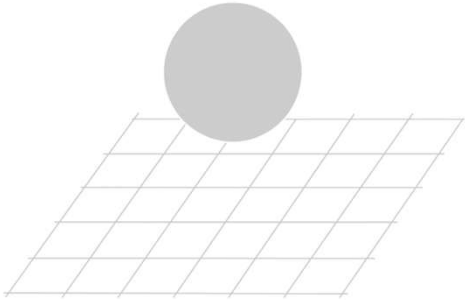
- This is in stark contrast with local QFT, where entropy is extensive

$$\mathcal{S}_{\text{QFT}} \propto L^{d-1} \sim \text{Volume}$$

- The area law is intimately related with the holographic principle, which limits the maximal amount of information in QG [Bekenstein '72, Bousso '99]

- However, the leading-order entropy is purely geometric and thus IR universal!

$$S_{2-\text{der}} = \frac{1}{2\kappa_d^2}\int d^dx\sqrt{-g}R + S_{\text{matter}} \quad \rightsquigarrow$$



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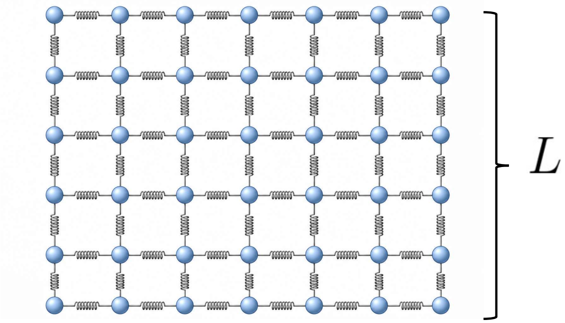

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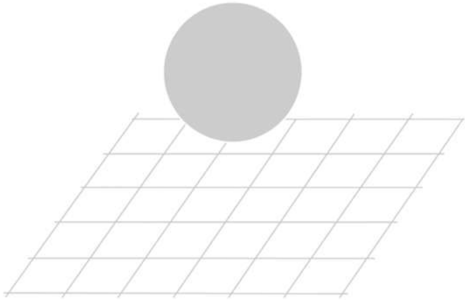

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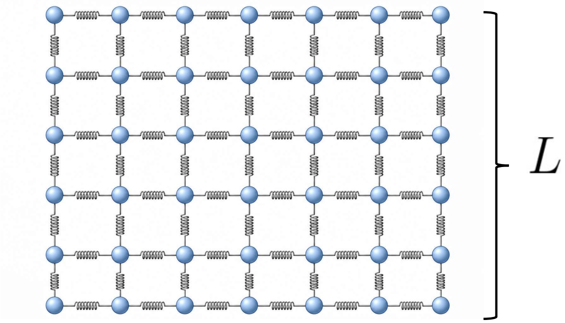

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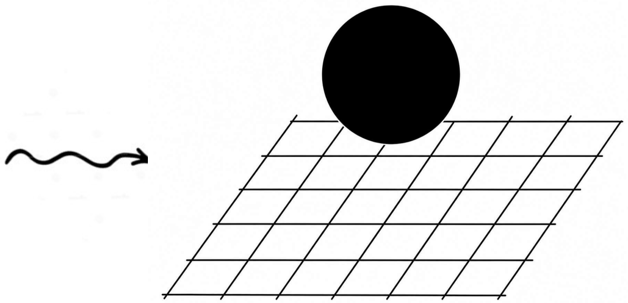

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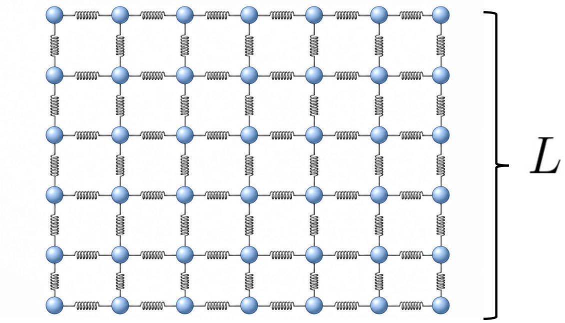

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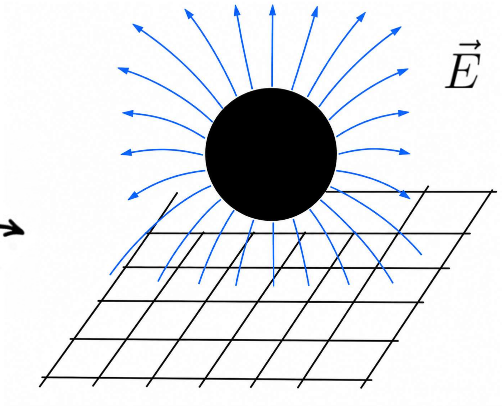

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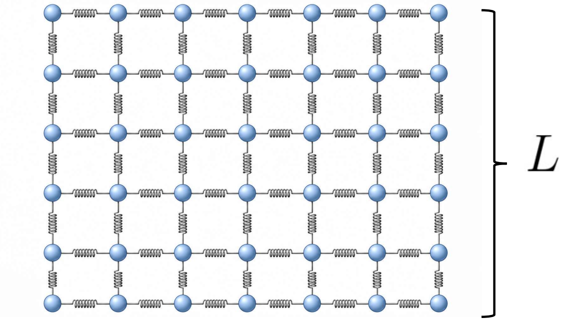

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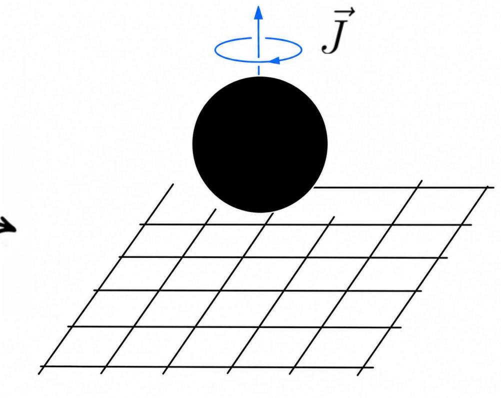

$$\boxed{\mathcal{S}_{\rm BH} = \frac{A_h}{4G_N}}$$

[Bekenstein '72; Hawking '75]

Black Hole Entropy Knows more about UV

- Thus, BHs probe QG DOFs but cannot distinguish between different UV completions…

$$S_{\mathrm{EFT,d}} = \frac{1}{2\kappa_d^2}\int \mathrm{d}^d x\sqrt{-g}\,R + (\mathrm{matter,\ boundary\ terms})$$

- Higher-derivative operators in EFT encapsulate non-trivial information about UV scales

… and this is reflected by the BH entropy [Wald '93]

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- Examining supersymmetric BHs shows agreement with Swampland expectations: [AC, Zatti '25]

1. *Perturbative corrections* encode when the EFT no longer describes correctly the BH entropy
2. *Minimal* BH entropy attained for 'quantum effective' areas of $\mathcal{O}(\Lambda_{\mathrm{QG}}^{-1})$ [see also Vafa '24]

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Higher-derivative terms

[AC, Herráez, Ibáñez '23;
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Perturbation Theory is not Enough!

- The BH entropy must have a statistical interpretation (microscopic counting)

$$\mathcal{S}_{\text{micro}} = k_B \ln \left[d(q^i) \right]$$

Microstates for a given macrostate

- However, one readily notices that Bekenstein-Hawking entropy is not quantized!
- E.g., take Type IIB on $K3 \times T^2$ and consider $\frac{1}{4}$-BPS Black Holes

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$Exp(\mathcal{S})$ not an integer…

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$$\underbrace{Q^2, \quad P^2, \quad Q \cdot P}_{\text{Electric-magnetic charges}} \quad \xrightarrow{\text{Higher-derivatives}} \quad \mathcal{S}_{\text{Wald}} = \pi\sqrt{\Delta} - 2\log\Delta + \mathcal{O}(1/\sqrt{\Delta})$$

$$\Delta := Q^2P^2 - (Q \cdot P)^2$$

$Exp(\mathcal{S})$ still not an integer...

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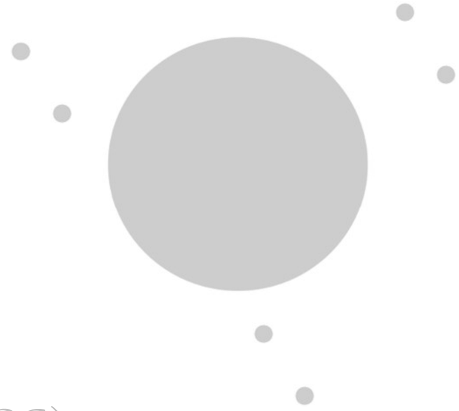
$$\underbrace{Q^2, \;\; P^2, \;\; Q \cdot P}_{\text{Electric-magnetic charges}} \quad \xrightarrow{\text{Non-Perturbative}} \quad \mathcal{S}_{\text{NP}} = \pi\sqrt{\Delta} - 2\log\Delta + \mathcal{O}(1/\sqrt{\Delta}) + \sum_{k=2}^{\infty} d_k \, e^{-\pi\sqrt{\Delta}/k} + \ldots = \mathcal{S}_{\text{micro}}$$

$Exp(\mathcal{S})$ gives an integer!

In This Talk...

- **Main goal**: Initiate systematic study of *(non-)perturbative* corrections to BH entropy
- We focus on SUSY theories and BPS objects $\rightsquigarrow$ Extremal BHs
- In particular, we consider 4d $\mathcal{N} = 2$ EFTs due to the richer set of *higher-derivative effects and massive spectra* exhibited by those



- The hope is to eventually be able to:

  1. Learn *new things* about interplay between BH entropy and UV completions (in QG)
  2. Test *robustness* of Swampland concepts/ideas under non-perturbative phenomena
  3. Use the quantum BH entropy as a probe of the *bulk of moduli space* [v. de Heisteeg, Vafa, Wiesner, Wu '22-'23]

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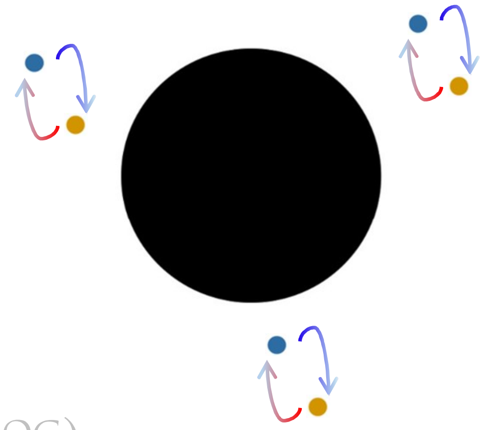

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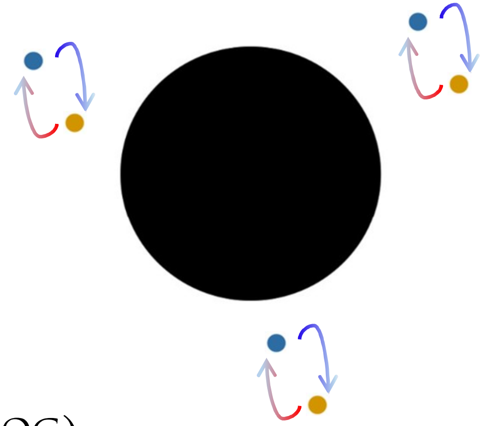

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1. Learn *new things* about interplay between BH entropy and UV completions (in QG)
2. Test *robustness* of Swampland concepts/ideas under non-perturbative phenomena
3. Use the quantum BH entropy as a probe of the *bulk of moduli space* [v. de Heisteeg, Vafa, Wiesner, Wu '22-'23]

Outline

I. An Exact Quantum Entropy Function
  i. Integrating out charged massive particles in $\mathrm{AdS}_2 \times S^2$
  ii. A close look @ non-perturbative effects

II. Applications in String Theory
  i. BPS black holes in 4d $\mathcal{N} = 2$ theories
  ii. Large volume approximation and D0-brane effects

III. Non-Perturbative Corrections as Complex Word-line Instantons
  i. A brief semiclassical analysis
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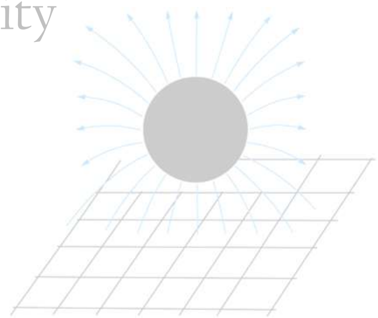
Part I

An Exact Quantum Entropy Function

Extremal BHs: Near-Horizon Geometry

- We want to study in detail the imprint of charged (BPS) matter in the quantum BH entropy
- For simplicity, consider first charged, static, extremal BH solutions of EM+Gravity



$$ds^2 = -f(r)^2 dt^2 + f(r)^{-2} dr^2 + r^2 d\Omega_2^2\,, \qquad f(r) = 1 - \frac{r_h}{r}$$

with $r_h = M = Q = \sqrt{Q_e^2 + Q_m^2}$.

- Zooming in to the near-horizon region, one finds some $\mathrm{AdS}_2 \times S^2$ geometry

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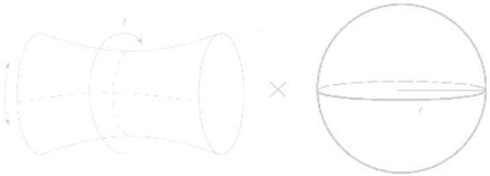

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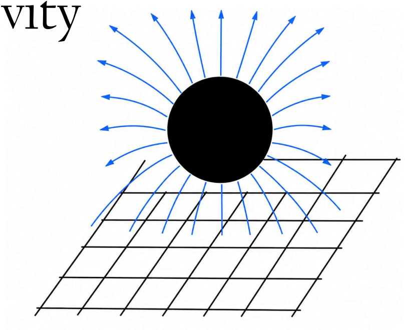

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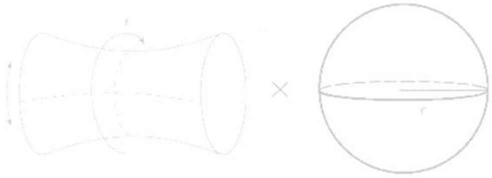

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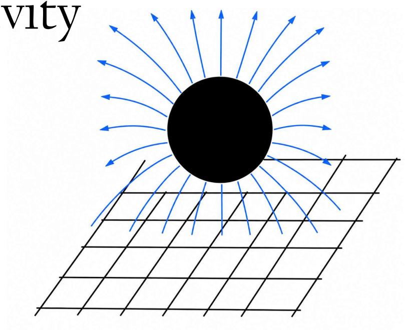

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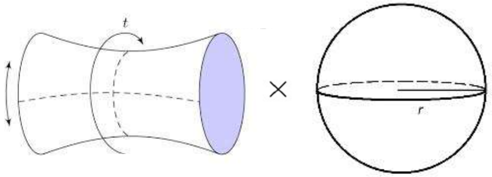

[Gibbons, Hull ’82; Gibbons, Maeda ‘88]

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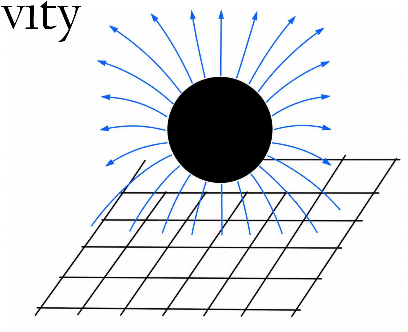

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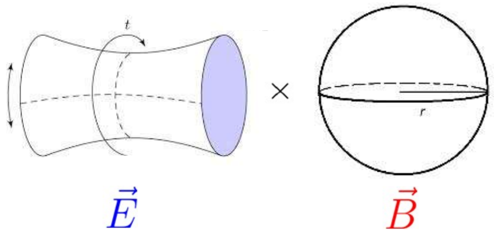

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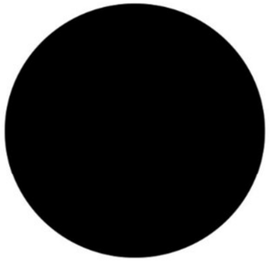
The Quantum Entropy Function

- Suppose our theory contains massive fields charged under the same *U(1)* as the BH
- **Question:** What is their effect on the BH entropy at the quantum level?
- According to [Sen '08] one must compute the effective action within the throat



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- For large charges, the saddle point approximation retrieves Wald's entropy formula

$$\mathcal{S}_{\text{Wald}} = 2\pi R^2 \left(Q_e E - \mathcal{L}_{\text{eff}}\right) = 2\pi R^2 \left(E\frac{\partial}{\partial E} - 1\right)\mathcal{L}_{\text{eff}}$$

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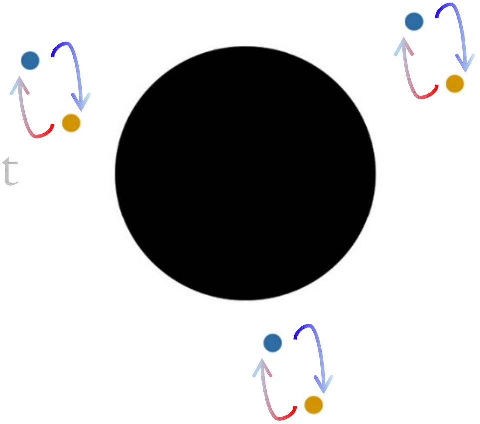

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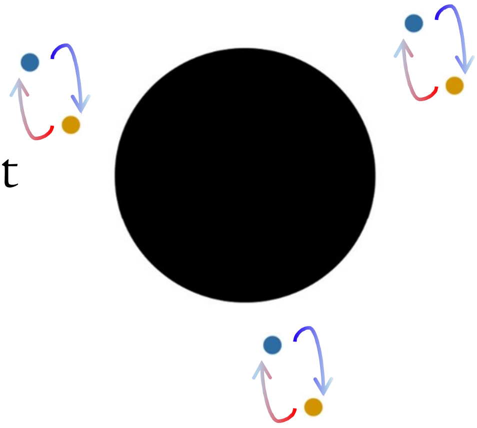

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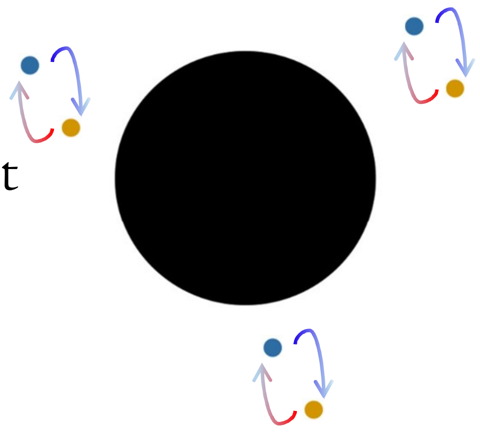

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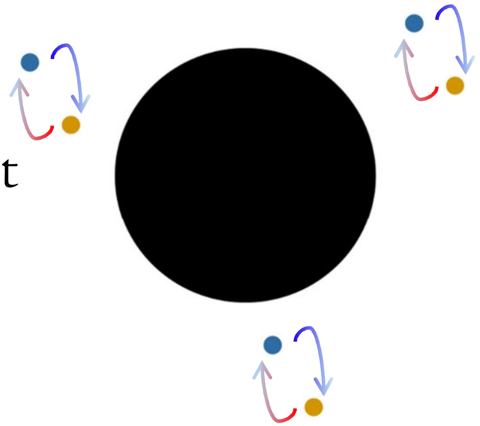

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Functional Determinants 101

- To integrate out minimally coupled fields, we use Schwinger proper-time formalism

- For bosons, we start from 4d quadratic action

$$S[\varphi,\phi] = S[\varphi] + \int d^4x\sqrt{-\det g}\,\phi^\dagger\left[-\mathcal{D}^2 - m^2\right]\phi \qquad \text{with} \quad \mathcal{D}^2 = -(\nabla - iA)^2$$

- The 1-loop path integral becomes a trace of the heat kernel operator associated with $\mathcal{D}^2$

$$\log \mathcal{Z}_\phi := -i\Delta\Gamma[\varphi] = -\int_0^\infty \frac{d\tau}{\tau}\, e^{-\frac{\epsilon^2}{4\tau}}\, e^{-\tau m^2}\mathrm{Tr}\left[e^{-\tau\mathcal{D}^2}\right]$$

- For fermions, the relevant (quadratic) action reads

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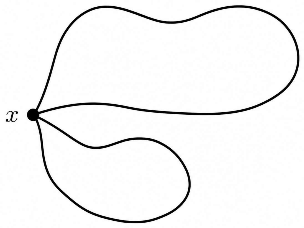

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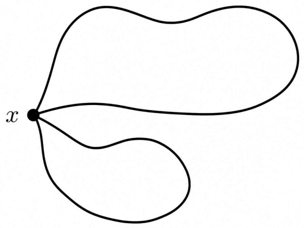

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The Full $\mathrm{AdS}_2 \times S^2$ Trace

- The previous path integrals can be solved exactly, i.e. non-perturbatively (in bckgd fields)!

- Consider the spin-0 case first. Splitting the operator in (commuting) AdS_2 and S^2 pieces:

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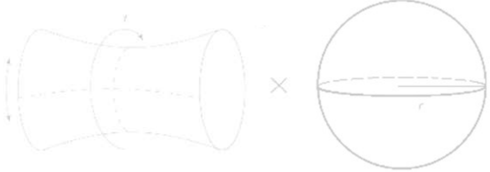

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- So far, this applies to any charged, extremal BH. For BPS 4d $\mathcal{N} = 2$ black holes one has

$$R^2_{\mathrm{AdS}} = |Z_{\mathrm{BH}}|^2\,, \quad q_e = \mathrm{Re}\left(\bar{Z}_{\mathrm{BH}} Z_{\mathrm{p}}\right), \quad q_m = \mathrm{Im}\left(\bar{Z}_{\mathrm{BH}} Z_{\mathrm{p}}\right) \qquad \text{with} \quad Z_p = e^{K/2}\left(p^A \mathcal{F}_A - q_A X^A\right)$$

- This implies that the path integral is purely real (i.e. no background instabilities)

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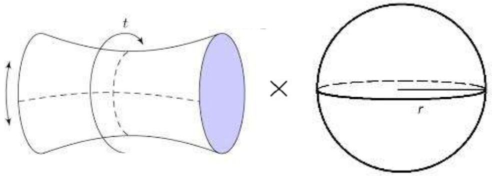

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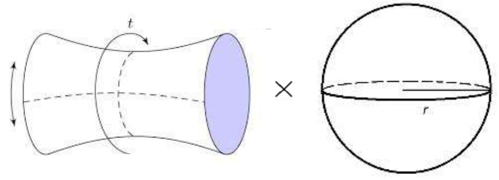

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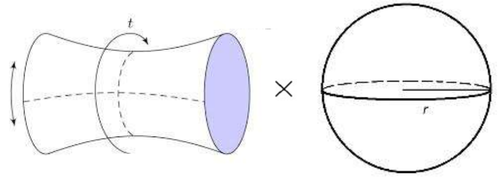

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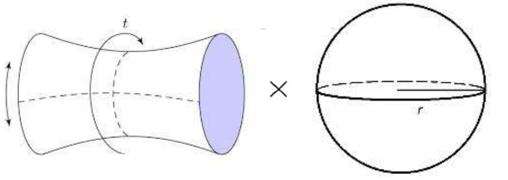

$E = e/R^2$ $\quad$ $B = g/R^2$

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- This implies that the path integral is purely real (i.e. no background instabilities)

$$m^2 \geq |Z_p|^2 \implies \Delta^2 \geq 0 \implies \log \mathcal{Z}_\phi \in \mathbb{R}$$

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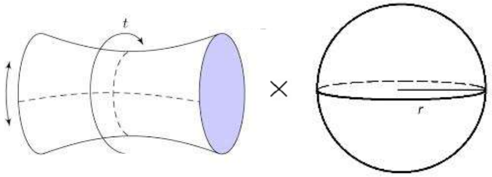

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- Now we'd like to put things together. The simplest susy combination is the Hypermultiplet

$$\text{Hypermultiplet} = 2 \times \text{Complex Scalars} + \text{Dirac Fermion}$$

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'Topological' term (phase of funct. Det.)

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Similar to QED!!

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- The 1st term is reminiscent of the 1-loop Schwinger effective action in QED

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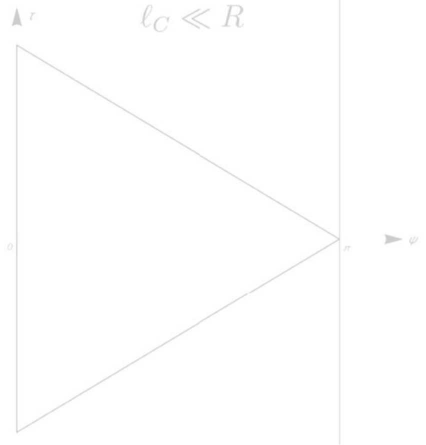
- The (complex) coupling β determines the background perceived by the charged matter
- Its modulus measures the ratio between Compton wavelength of the particle and AdS radius

$$|\beta^{-1}| = mR \quad \rightsquigarrow \quad |\beta| \ll 1 \qquad \text{weak coupling}$$



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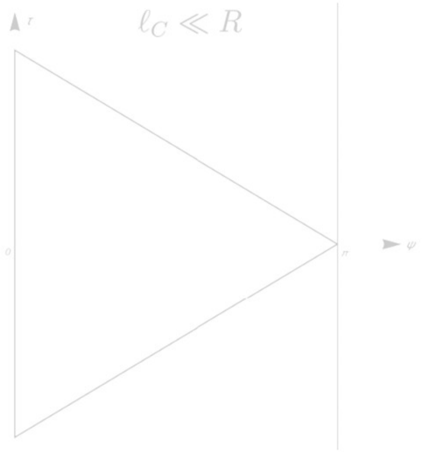

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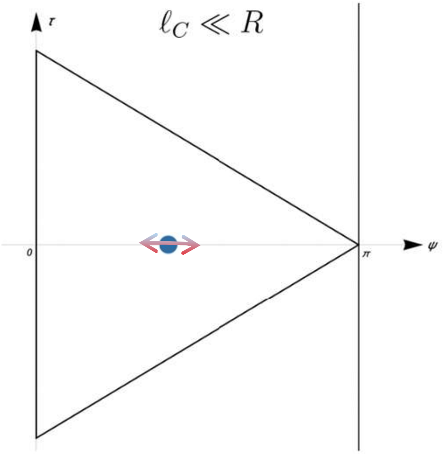

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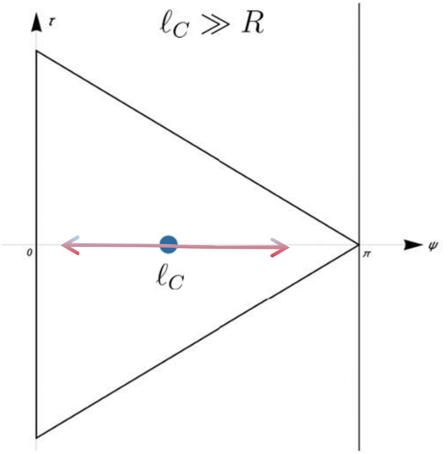

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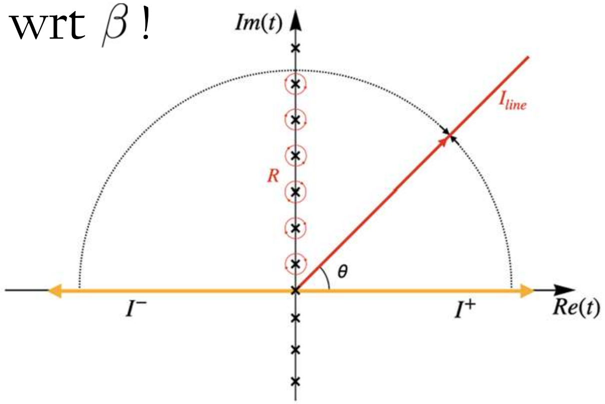

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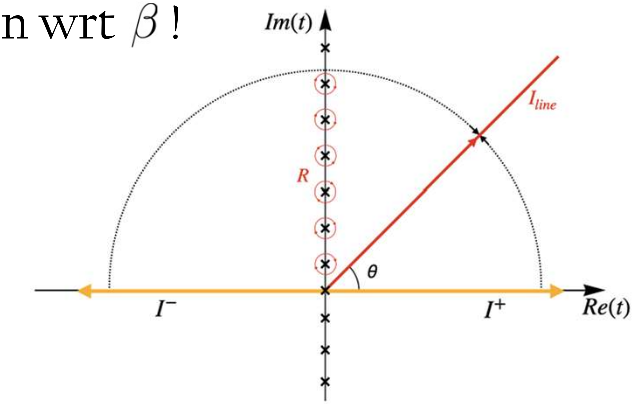

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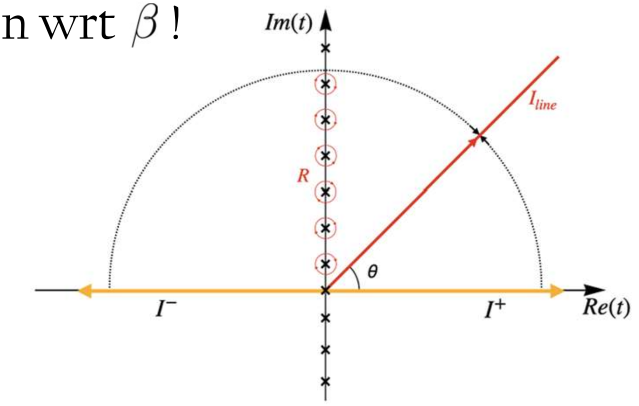

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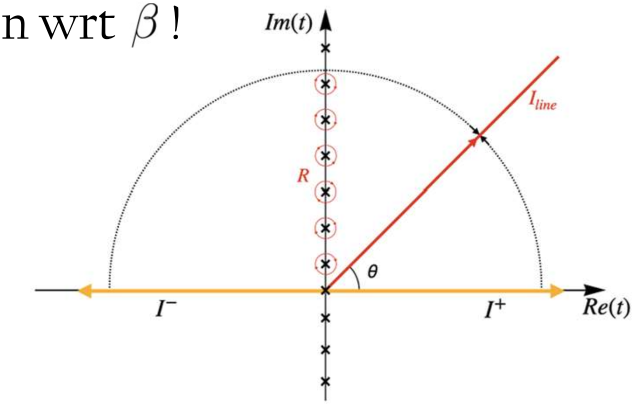

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Non-Perturbative in β

A Few Questions & Remarks

- The non-perturbative terms exhibit a similar structure than AdS$_2$ wordline instantons

$$\mathcal{R}_{\rm AdS} \propto i \left[\mathrm{Li}_2\left(-e^{-2\pi(q_e+\sqrt{q_e^2-m^2})}\right) + 2\pi\sqrt{q_e^2-m^2}\,\mathrm{Li}_1\left(-e^{-2\pi(q_e+\sqrt{q_e^2-m^2})}\right)\right]$$

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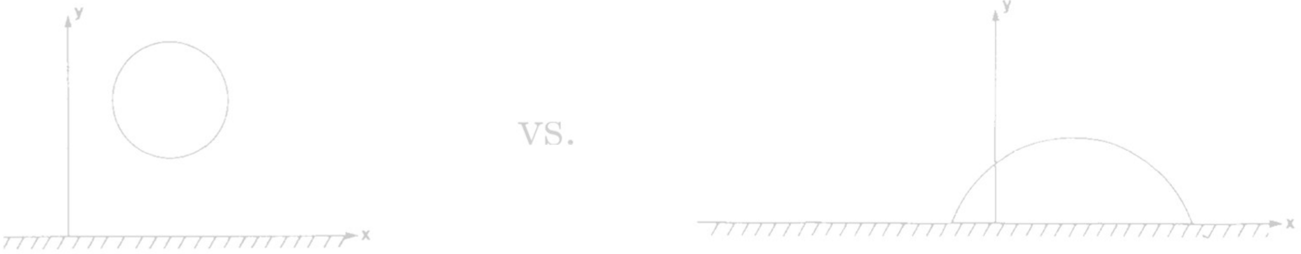

- What precise info do they capture? Do they account alone for all NP charged matter effects?

- Two special cases with no NP effects: purely electric ($B=0$) and purely magnetic ($E=0$)

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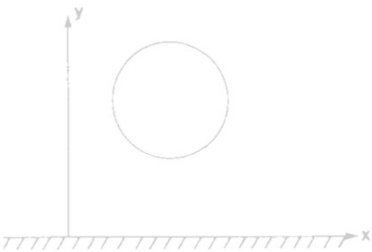

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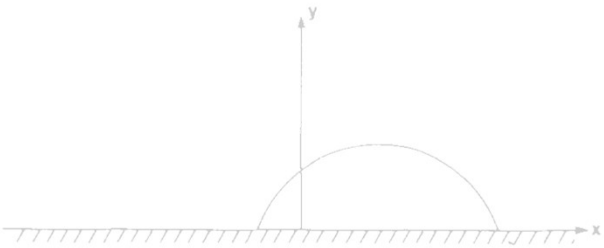

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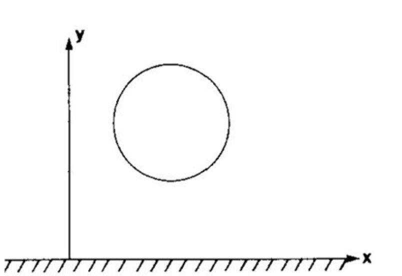

Superextremal ($q_e > mR$)

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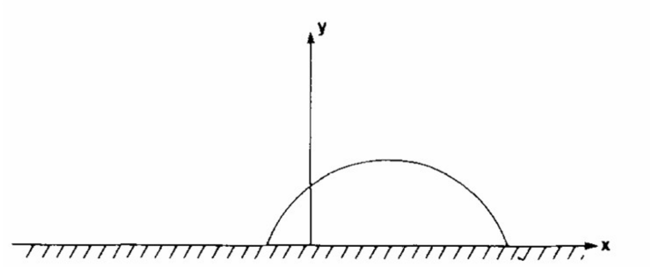

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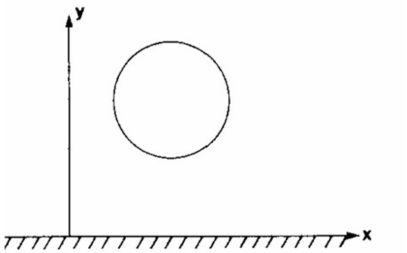

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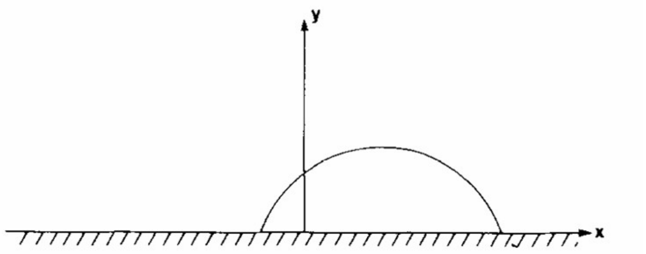

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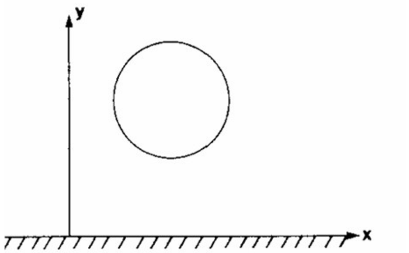

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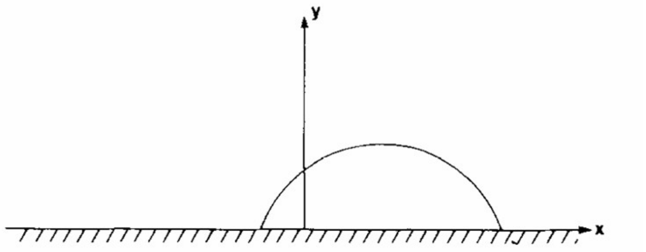

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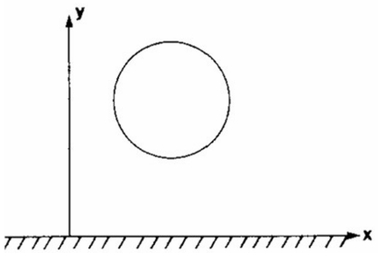

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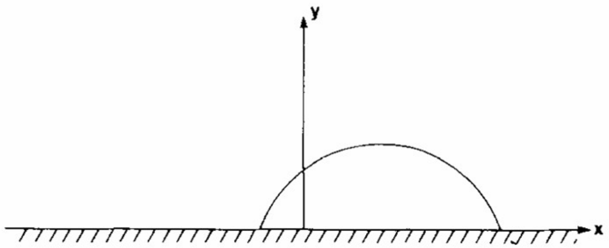

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Outline

I. An Exact Quantum Entropy Function

  i. Integrating out charged massive particles in $\mathrm{AdS}_2 \times S^2$

  ii. A close look @ non-perturbative effects

II. Applications in String Theory

  i. BPS black holes in 4d $\mathcal{N} = 2$ theories

  ii. Large volume approximation and D0-brane effects

III. Non-Perturbative Corrections as Complex Word-line Instantons

  i. A brief semiclassical analysis

  ii. Novel complex instantons

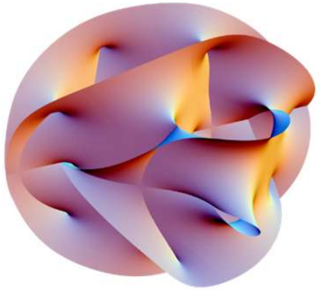
IV. Summary and Outlook

Part II

Applications in String Theory

4d N=2 Theories: The Lagrangian

- Take Type IIA on CY 3-fold $\rightsquigarrow$ 4d theories preserving 8 supercharges
- Two-derivative (bosonic) action



$$\begin{aligned} S_{\rm IIA}^{\rm 4d} &= \frac{1}{2\kappa_4^2}\int \mathcal{R}\star 1 + \frac{1}{2}\mathrm{Re}\,\mathcal{N}_{AB}F^A\wedge F^B + \frac{1}{2}\mathrm{Im}\,\mathcal{N}_{AB}F^A\wedge\star F^B \\ &\quad -\frac{1}{\kappa_4^2}\int G_{a\bar b}\,dz^a\wedge\star d\bar z^b + h_{pq}\,dq^p\wedge\star dq^q\,, \end{aligned}$$

- The vector multiplet sector is completely determined by the prepotential

$$\mathcal{F} = \frac{1}{2}X^A\mathcal{F}_A\,, \qquad \text{where} \quad \mathcal{F}_A = \partial_{X^A}\mathcal{F}$$

$$G_{a\bar b} = \partial_a\partial_{\bar b}K\,, \qquad \text{with} \quad K = -\log i\left(\bar X^A\mathcal{F}_A - X^A\bar{\mathcal{F}}_A\right)\,, \qquad z^a = \frac{X^a}{X^0}$$

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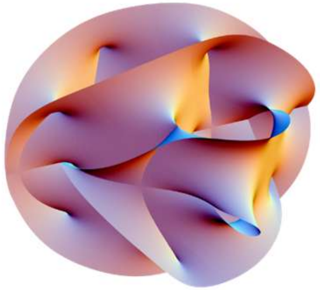

VM

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HM

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4d N=2 Theories: Higher-derivatives

- Beyond 2-derivatives, there are interesting higher-dimensional BPS operators

[Bershadsky, Cecotti, Ooguri, Vafa '94
Antoniadis, Gava, Narain, Taylor '95]

$$\mathcal{L}_{\text{h.d.}} \supset -\frac{i}{2}\sum_{g\geq 1}\int \mathrm{d}^4\theta\,\mathcal{F}_g(\mathcal{X}^A)\,\left(\mathcal{W}^{ij}\mathcal{W}_{ij}\right)^g \;+\; \text{h.c.} \supset -\frac{i}{2}\sum_{g\geq 1}\mathcal{F}_g(X^A)\,\mathcal{R}_-^2\,W_-^{2g-2} \;+\; \text{h.c.}$$

Scalars

Graviton & graviphoton (anti-self-dual)

- They can be encoded into the generalized prepotential [Ooguri, Vafa, Strominger '04]

$$F(X,W^2)=\sum_{g=0}^{\infty}F_g(X^A)W^{2g} \qquad \text{with} \quad F_g(X^A)=(-1)^g 2^{-6}\mathcal{F}_g(X^A)$$

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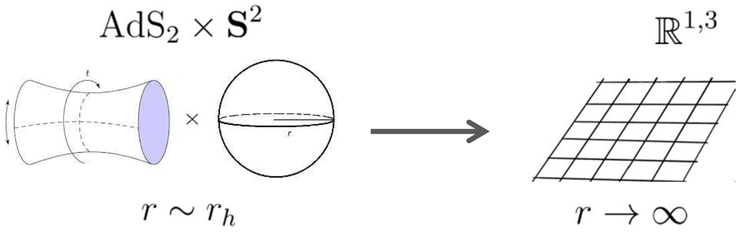
4d N=2 BPS Black Holes



- These theories admit BPS (extremal) black hole solutions

$$ds^2 \simeq -\frac{r^2}{r_h^2}\,dt^2 + \frac{r_h^2}{r^2}\left(dr^2 + r^2 d\Omega_2^2\right) \qquad \Upsilon \propto W^2 = \text{const.}$$

- The effective lagrangian giving the entropy function is the gen. prepotential!

[Lopes-Cardoso, Wit, Mohaupt '98-'99]

$$\mathcal{S}_{\mathrm{BH}} = \pi\left[|\mathscr{Z}|^2 + 4\mathrm{Im}\,\left(\Upsilon\partial_\Upsilon F(Y,\Upsilon)\right)\right] = 4\pi\left[1 - \mathrm{Re}\,Y^A\frac{\partial}{\partial \mathrm{Re}\,Y^A}\right]\mathrm{Im}\,F$$

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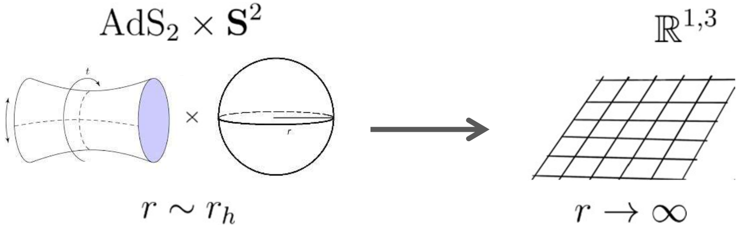

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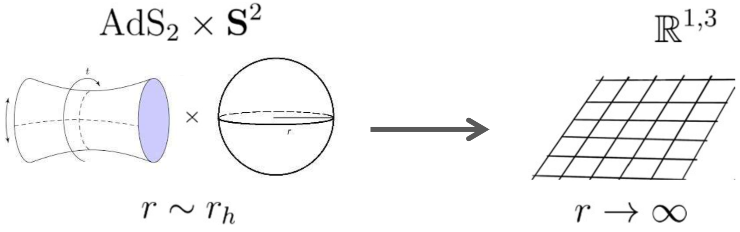

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Bekenstein-Hawking

Deviations from area law

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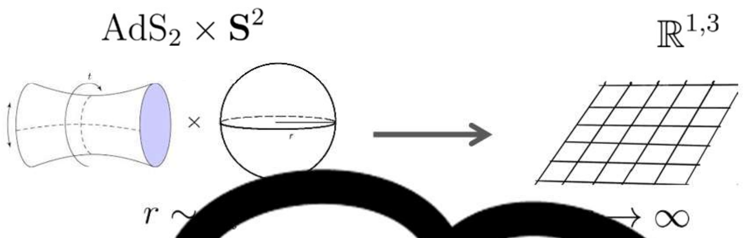

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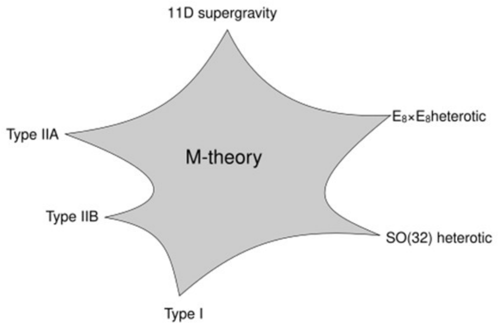
Large Volume Regime & D0-branes



- To make contact with our previous discussion, we consider the LVR

$$\text{Type IIA on CY}_3 \Longleftrightarrow \text{M-theory on CY}_3 \times \mathbf{S}^1$$

- The lightest (tower of) states correspond to D0-branes. How do they contribute to $\operatorname{Im} F$?

$$\mathcal{I} = \sum_{n \in \mathbb{Z}} \mathcal{I}_n \qquad \text{with} \quad \mathcal{I}_n = \operatorname{Re}\left[\int_{0+}^{\infty} \frac{dt}{t} \frac{e^{-it\beta_n^{-1}}}{\sinh^2(t/2)}\right]$$

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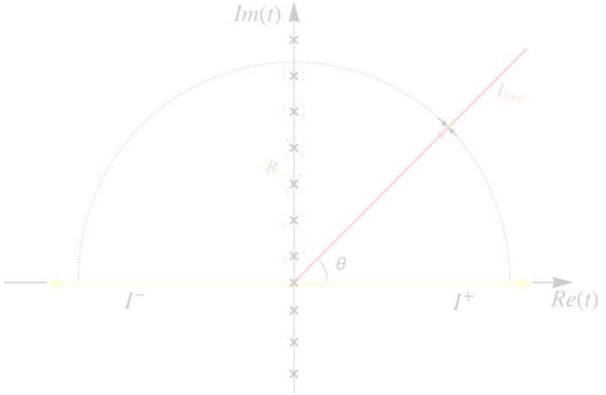

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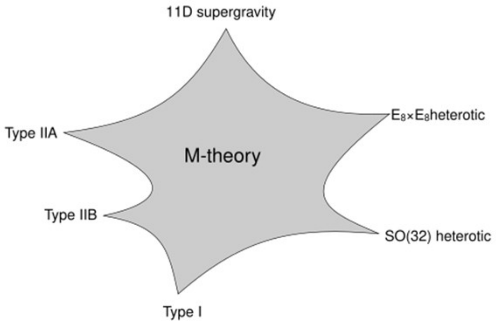

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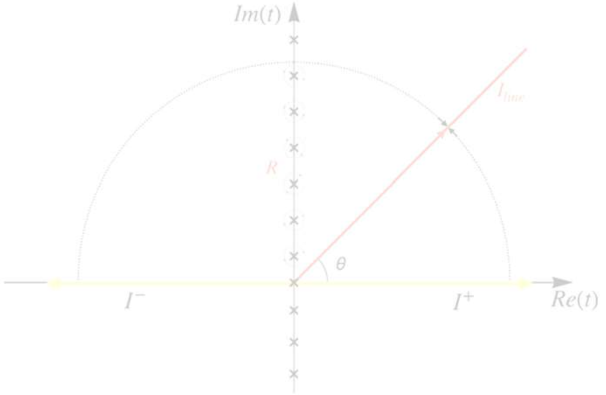

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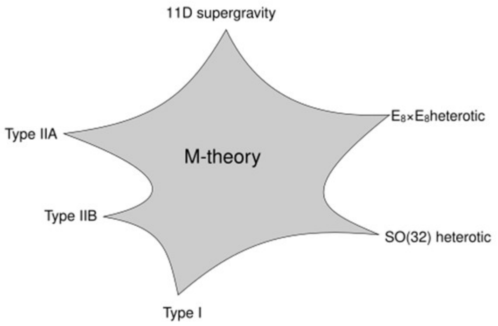

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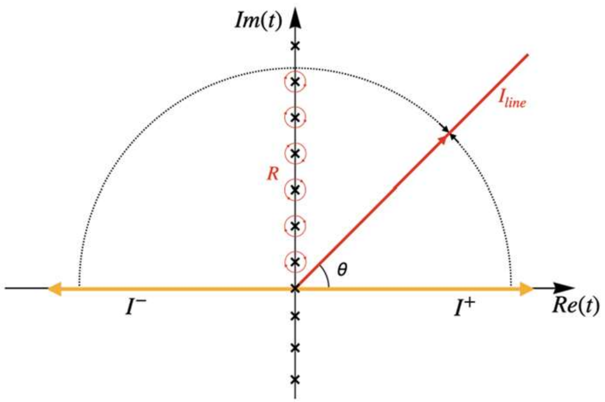
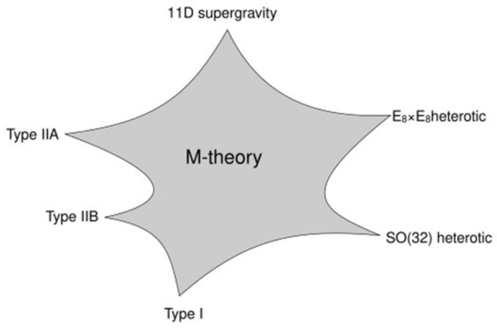


[Gopakumar, Vafa '98]

Match with Gopakumar-Vafa!!

Large Volume Regime & D0-branes



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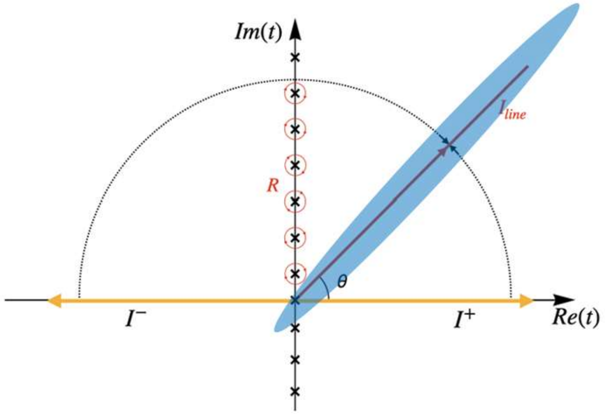
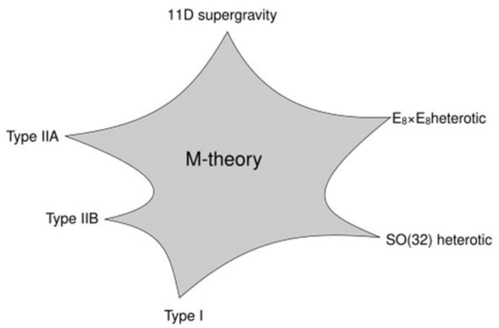


$$\mathcal{I}_{\text{line}}(\alpha) = \mathrm{Re}\left\{\sum_{k=1}^{\infty} \frac{1}{k \sinh^2\left(\frac{\alpha k}{2}\right)}\right\}$$

Perturbative in α

Large Volume Regime & D0-branes



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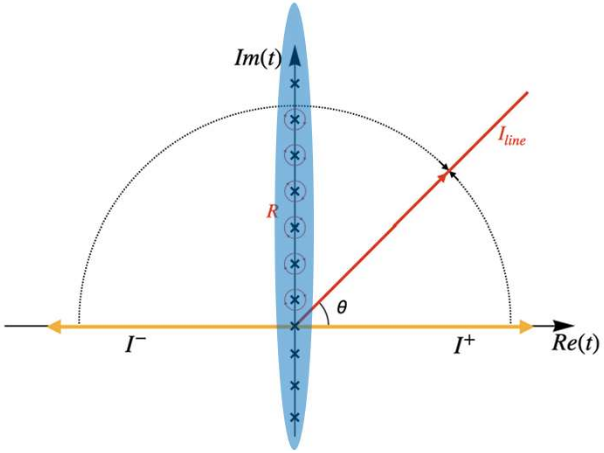

$$\mathcal{R}(\alpha) = \mathrm{Im}\left\{\frac{2}{\pi}\sum_{n\geq 1}\left[\mathrm{Li}_2\left(e^{-\frac{4\pi^2}{\alpha}n}\right) + 2\pi n\alpha\, \mathrm{Li}_1\left(e^{-\frac{4\pi^2}{\alpha}n}\right)\right]\right\}$$

Non-perturbative in α

Complexified Perturbative Coupling

- The perturbative coupling α allows us to expand the prepotential at large volume

$$F(Y,\Upsilon) = \frac{D_{abc}Y^aY^bY^c}{Y^0} + d_a \frac{Y^a}{Y^0}\Upsilon + G(Y^0,\Upsilon) + \mathcal{O}\left(e^{2\pi i z^a}\right)$$

Tree level

g = 1 corrections

(R^2 corrections)

Higher-genus corrections

(R^2W^{2g-2} corrections)

- Leading quantum correction (due to constant maps) given (for $|\alpha| \ll 1$) by

$$G(Y^0,\Upsilon) = -\frac{i}{2(2\pi)^3}\chi_E(X_3)\,(Y^0)^2 \sum_{g=0,2,3,\ldots} c^3_{g-1}\,\alpha^{2g} + \ldots$$

$$\text{with} \quad c^3_{g-1} = (-1)^{g-1}2(2g-1)\frac{\zeta(2g)\zeta(3-2g)}{(2\pi)^{2g}}\,, \qquad \alpha^2 = -\frac{1}{64}\frac{\Upsilon}{(Y^0)^2}$$

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Asymptotic growth

Expansion parameter

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$$\mathcal{L}_{\text{h.d.}} \supset \sum_{g\geq 1} m_{\text{D0}}^{2-2g}\,\mathcal{R}_-^2\,W_-^{2g-2}$$

Complexified Perturbative Coupling

- Physically, it measures the size of BH wrt dual Kaluza-Klein circle (also related to top. $g_s \in \mathbb{C}$)

$$|\alpha| = \frac{r_5}{r_h}\,, \qquad r_h = |Z_{\rm BH}|M_p^{-1}\,, \qquad (r_5)^{-1} = m_{\rm D0} = |Z_{\rm D0}|M_p$$

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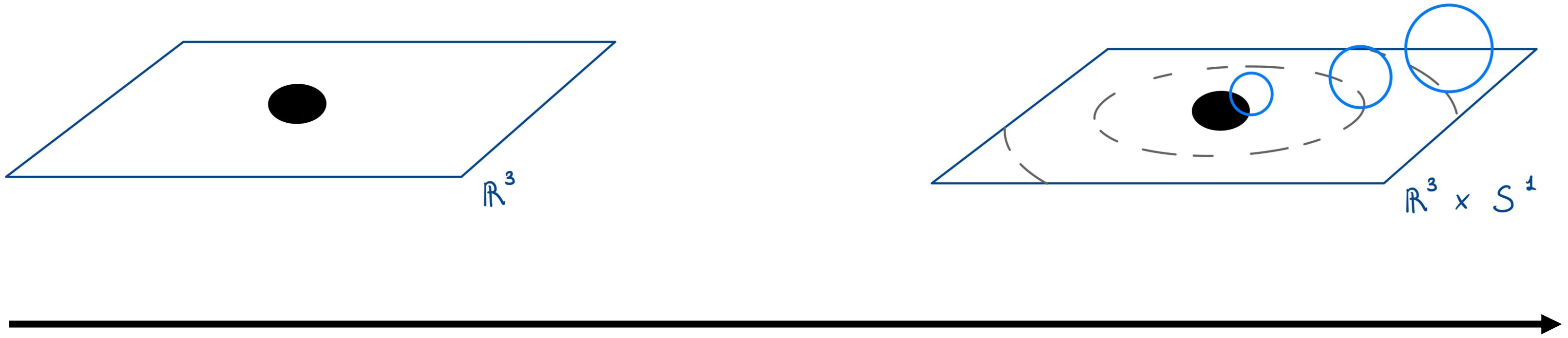

BH in 4d

$\alpha \ll 1$

$r_h \sim r_{D0}$

$\alpha \sim 1$

Outline

I. An Exact Quantum Entropy Function

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  ii. A close look @ non-perturbative effects

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  i. A brief semiclassical analysis

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IV. Summary and Outlook

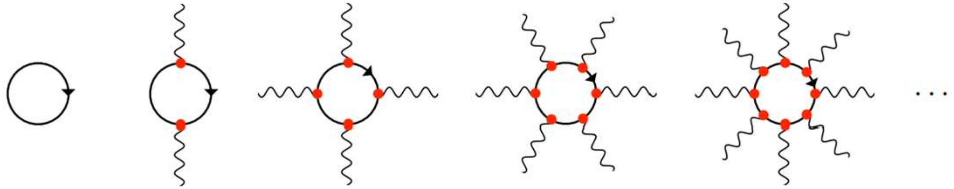
Part III

Non-Perturbative Corrections as Complex World-line Instantons

Euclidean Instantons in QFT

- In ordinary QFT, some backgrounds become unstable under quantum non-perturbative effects
- A celebrated example is QED in 4d

$$\Gamma^{1-\text{loop}}_{\text{sQED}}[A_\mu] = i \log\det(-\mathcal{D}^2 - m^2) \qquad \sim$$



- However, the QED action (for constant $\vec{E}$) has an imaginary part invisible to pert. theory

$$\text{Im}\,\mathcal{L}[E] = \frac{e^2E^2}{16\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\exp\left[-\frac{m^2\pi n}{eE}\right]$$

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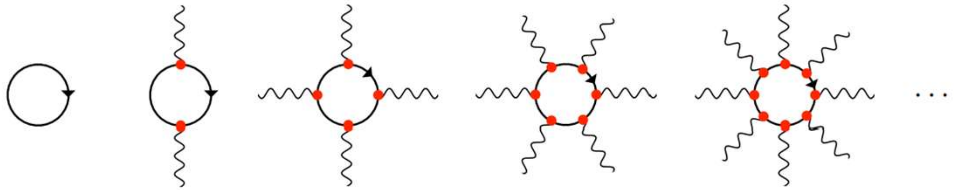

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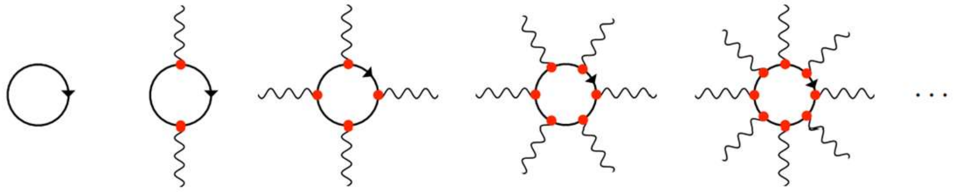

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- This can be calculated exactly… or approximated using semiclassical wordline instantons

Euclidean Instantons in QFT

- The idea is to rewrite the 1-loop determinant as a wordline path integral

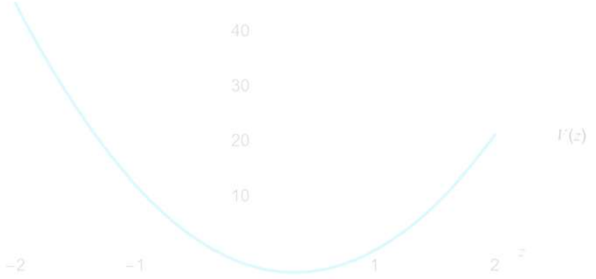
$$\Gamma^{1-\text{loop}}_{\text{sQED}}[A_\mu] = i \log \det\left(-\mathcal{D}^2 - m^2\right) = -i \int_{0+}^{\infty} \frac{d\tau}{\tau}\, e^{im^2\tau} \int d^4x \oint Dx\, e^{iS[x^\mu]}$$

- We then Wick rotate time $t \to -it_E$, and perform a semiclassical saddle expansion

$$\Gamma^{1-\text{loop}}_{\text{sQED}}[A_\mu] \simeq \sum_{i \in \text{saddles}} \mathcal{P}_i\, e^{iS[x_i^\mu]}$$

- The trivial saddle (point) gives back perturbative piece. Crucially, there exist more saddles

$$\dot{z}^2 - (p_t - q_e z)^2 + m^2 = 0$$



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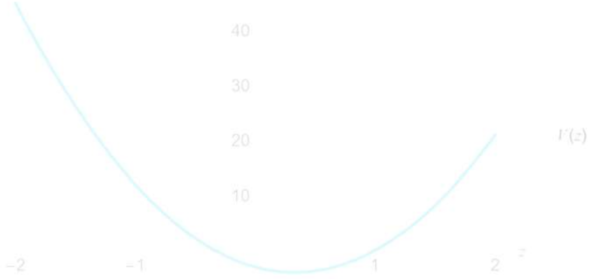

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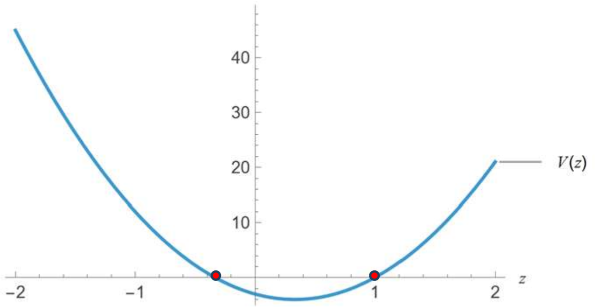
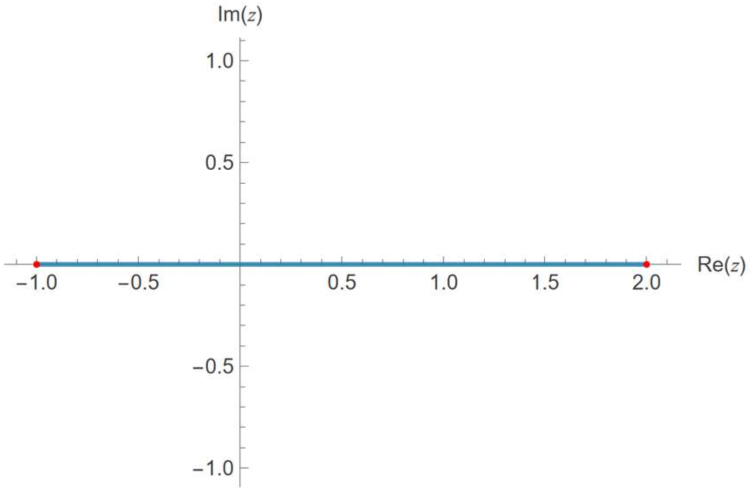
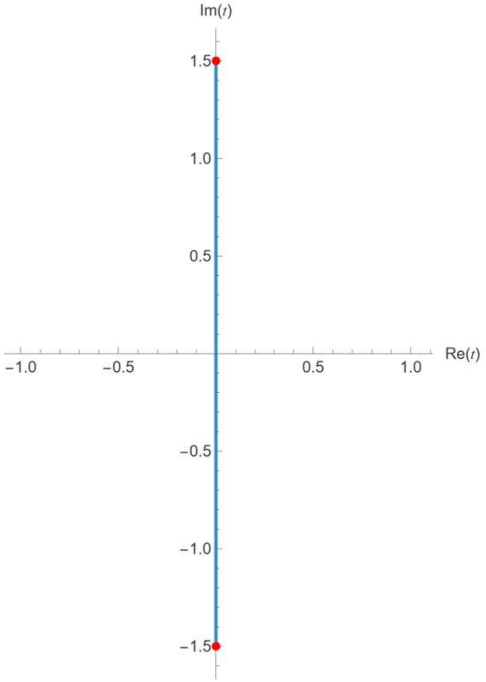
Conserved Energy



Euclidean Instantons in QFT

- One can easily find bouncing solutions between turning points

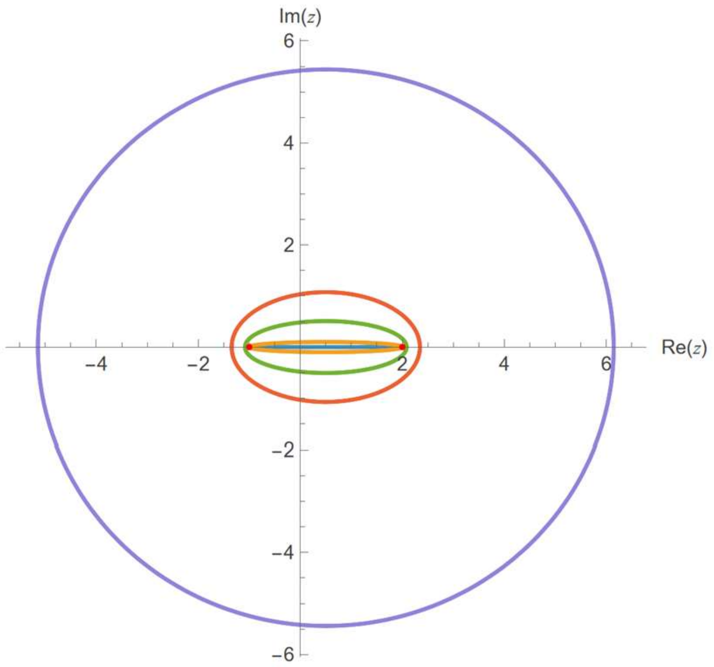




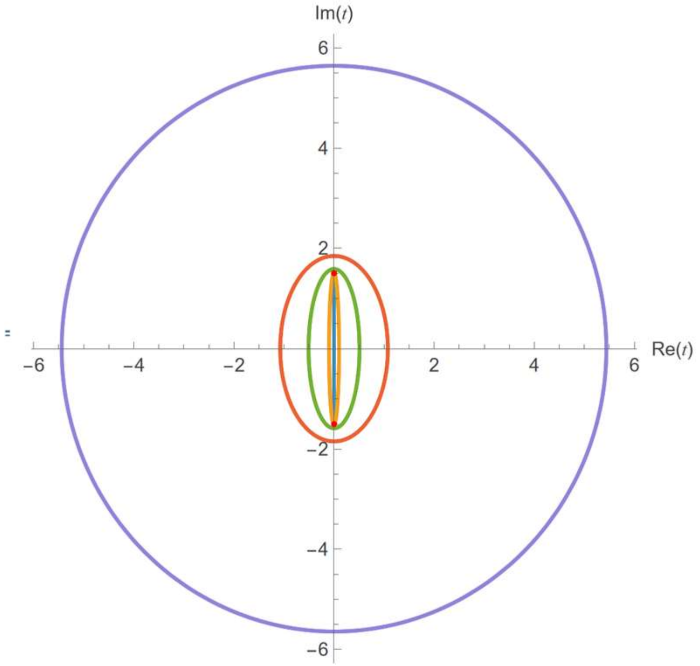
- They become real sols in Euclidean space (t_E, z) with action $iS^{(n)}_{\text{inst.}} = -\pi n m^2/q_e$

Euclidean Instantons in QFT

- The Euclidean formalism however falls short! One actually needs to look for complex instantons



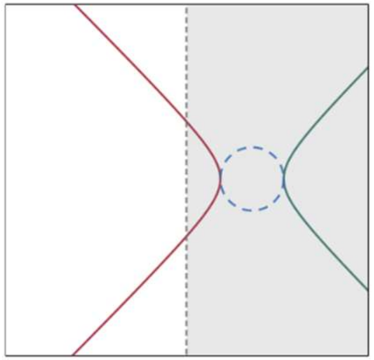
$$z(u) = z_0 + i\frac{m}{q_e}\,\sin\left(\frac{q_e}{m}(u+u_0)\right)$$



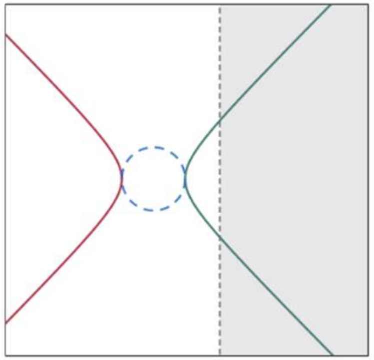
$$t(u) = t_0 + i\frac{m}{q_e}\,\cos\left(\frac{q_e}{m}(u+u_0)\right)$$

Geodesic Motion in Euclidean AdS_2

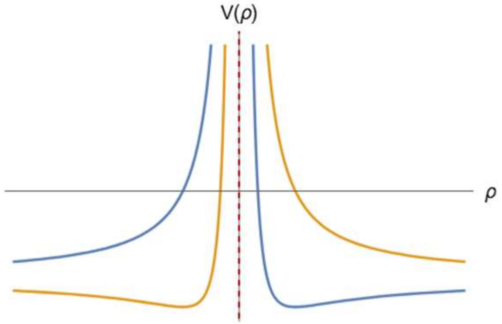
- The on-shell paths are circles in Euclidean AdS_2 [Pioline, Troost '05; **AC**, Lust, Montella, Zatti '25]



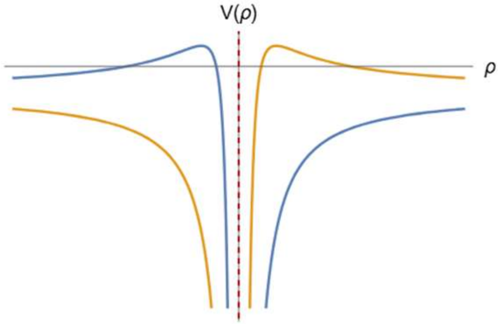
superextremal (yellow)



superextremal (blue)



$q_e^2 < \tilde{m}^2 + \ell^2$ (subextremal)



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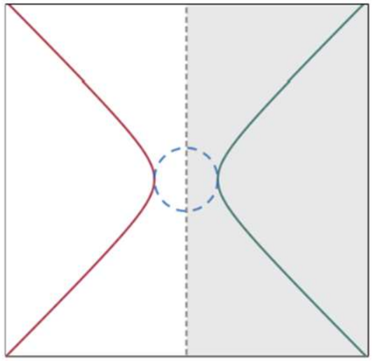

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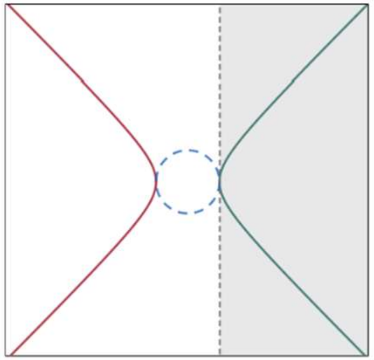
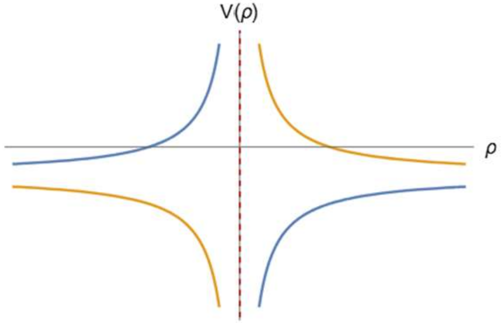


extremal



$q_e^2 = \tilde{m}^2 + \ell^2$ (extremal)

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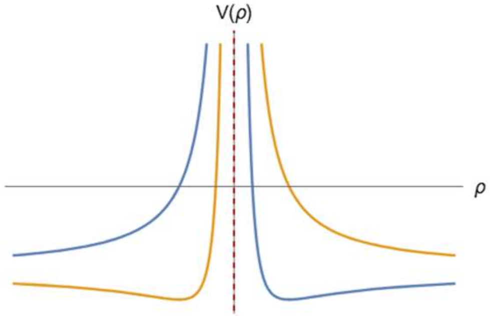

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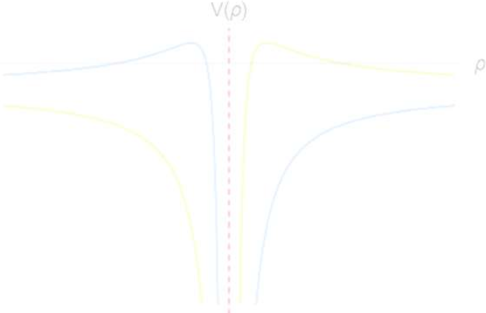

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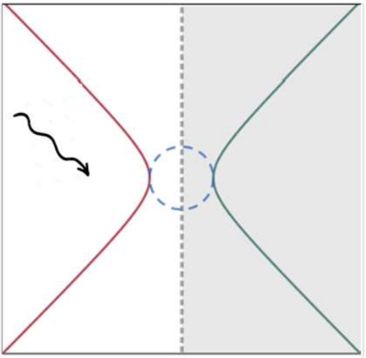
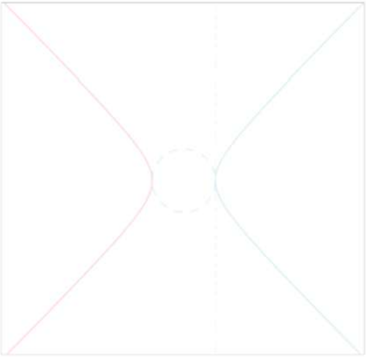
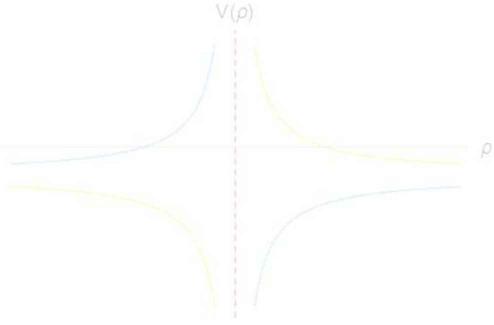
∞ Action?



subextremal



extremal



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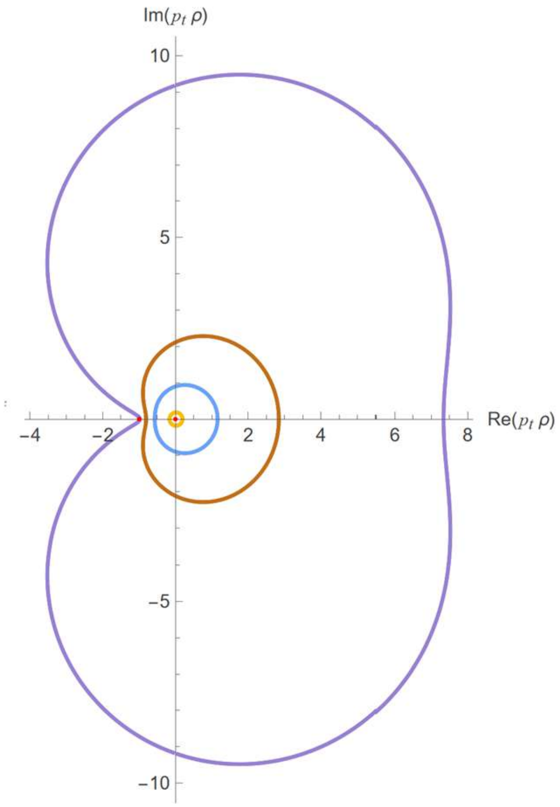
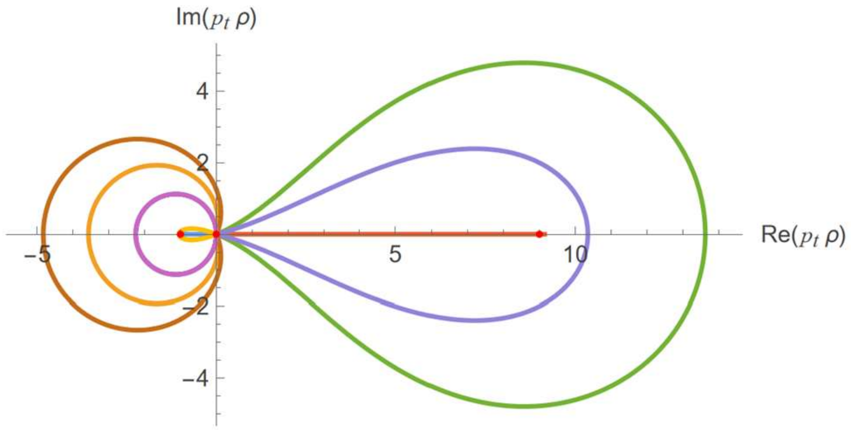
Complex Worldline Instantons in AdS$_2$

- One finds different behaviors in real/imaginary proper time [AC, Chu, Law-Smith '26]



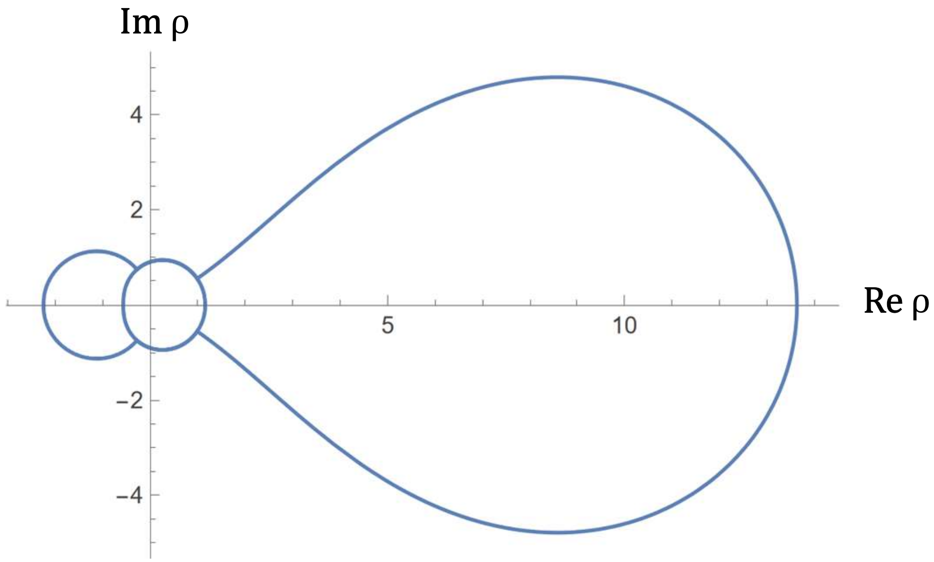
$\boldsymbol{\tau} \in \mathbb{R}$



$\boldsymbol{\tau} \in i\,\mathbb{R}$

Complex Worldline Instantons in AdS$_2$

- Combining the ‘instanton’ and ‘interference’ phases one finds novel ℂ sols [AC, Chu, Law-Smith ‘26]



- Their action is finite and reproduces exactly the non-perturbative residues!

$$\mathcal{R}(\alpha) \supset -\frac{\chi E}{16\pi^2}\sum_{n,k=1}^{\infty}\frac{q_m^{(n)}}{k}\,\mathrm{Re}\left[e^{-2\pi^2 k\left(q_e^{(n)}+iq_m^{(n)}\right)}\right]$$

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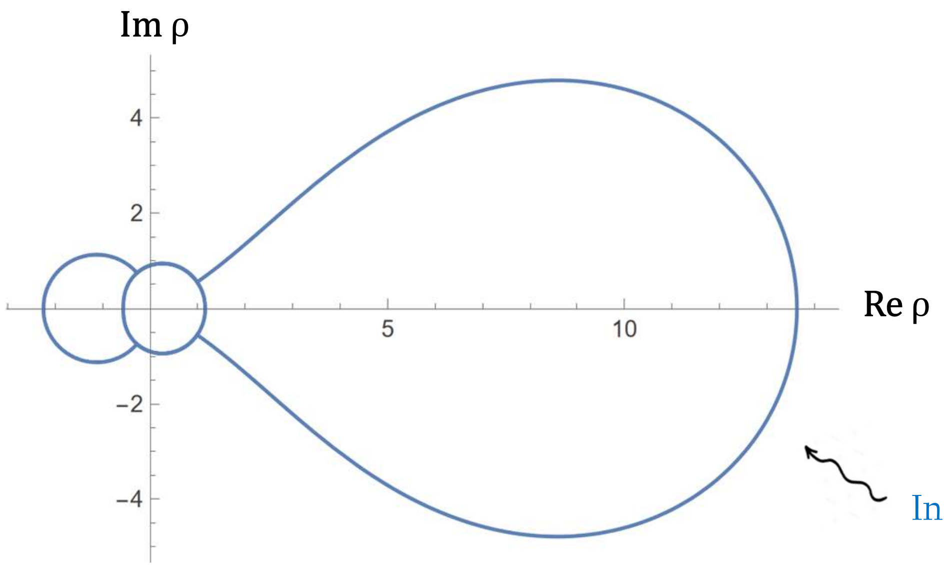

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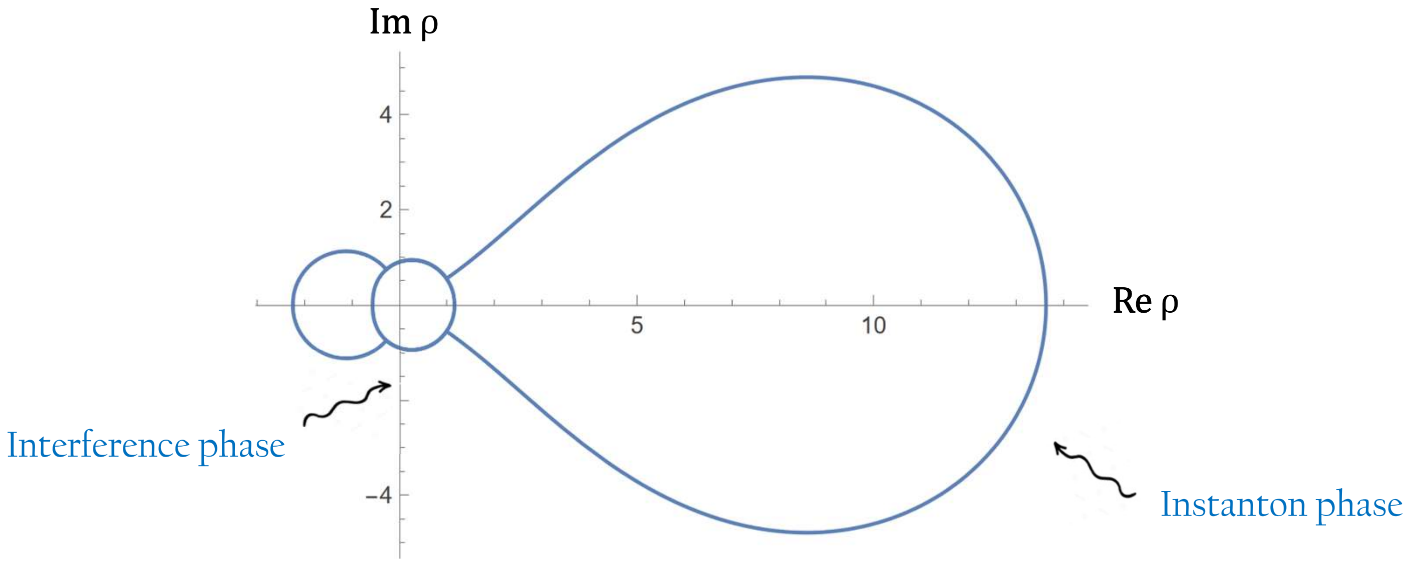

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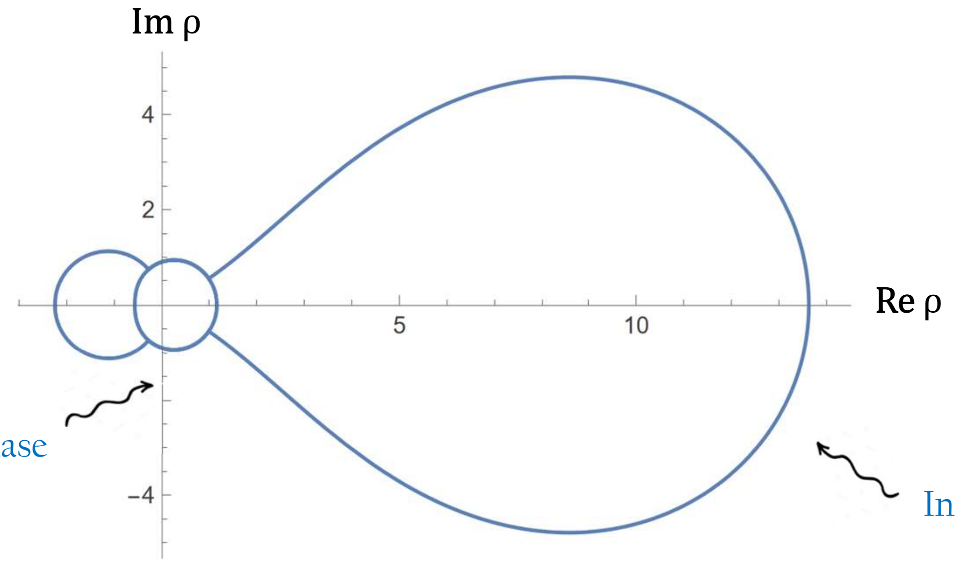

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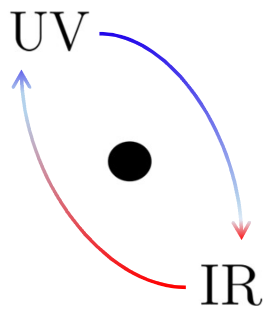
Part IV

Summary and Outlook

Summary & Outlook

- We have initiated a close investigation of (non-)perturbative corrections to BH entropy due to charged BPS matter



- We link the NP structure to the physics of charged probe branes
  1. The BPS probe is exactly *electric* (D0-D2-D4)
  2. The BPS probe is exactly *magnetic* (D2-D6 + symp. duals)

⇝ NP Effects turned off

- Perturbation theory is organized in terms of complexified coupling $\beta^{-1} = (q_e + i q_m) R^2$
- The latter measures how strongly the probes feel the grav. and EM backgrounds
- Thus, it also determines when (non-)perturbative quantum corrections become important
- In the semiclassical theory, the NP effects can be reproduced by novel complex saddles!

Summary & Outlook

- There are many possible extensions of our work
  1. Going beyond large volume (e.g. include WS instantons)
  2. BHs probing the F-theory limit in elliptic CYs [WIP]
  3. BHs probing weakly coupled string phases [WIP]
  4. Small BHs [WIP]
- Include the effect of higher spin fields via exact functional determinants [WIP]
- Extension to other backgrounds besides flat spacetime
- Stay tuned!

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LEINWEBER INSTITUTE FOR THEORETICAL PHYSICS

Kavli Institute
for Cosmological Physics
at the University of Chicago

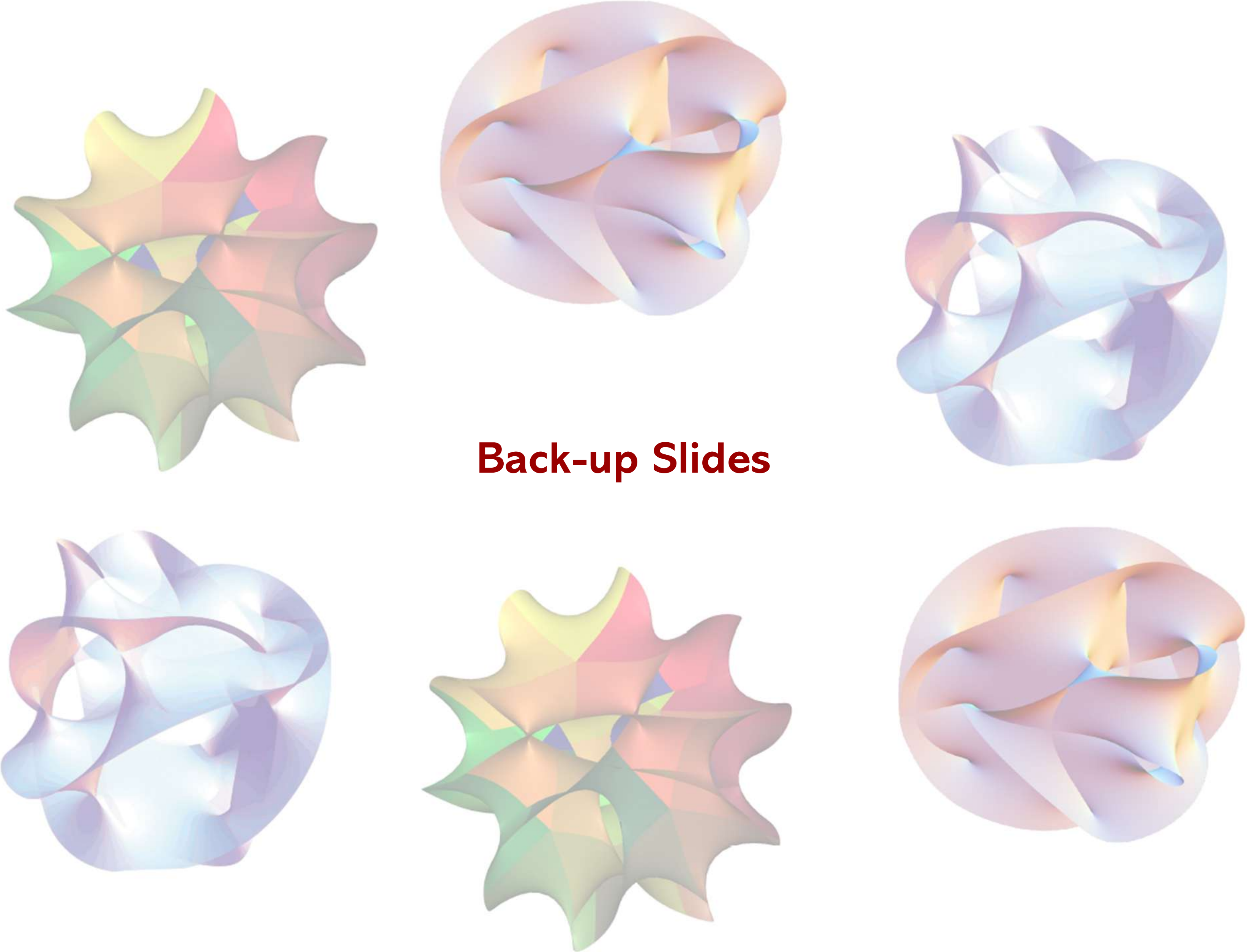
ENRICO FERMI INSTITUTE

THE UNIVERSITY OF CHICAGO

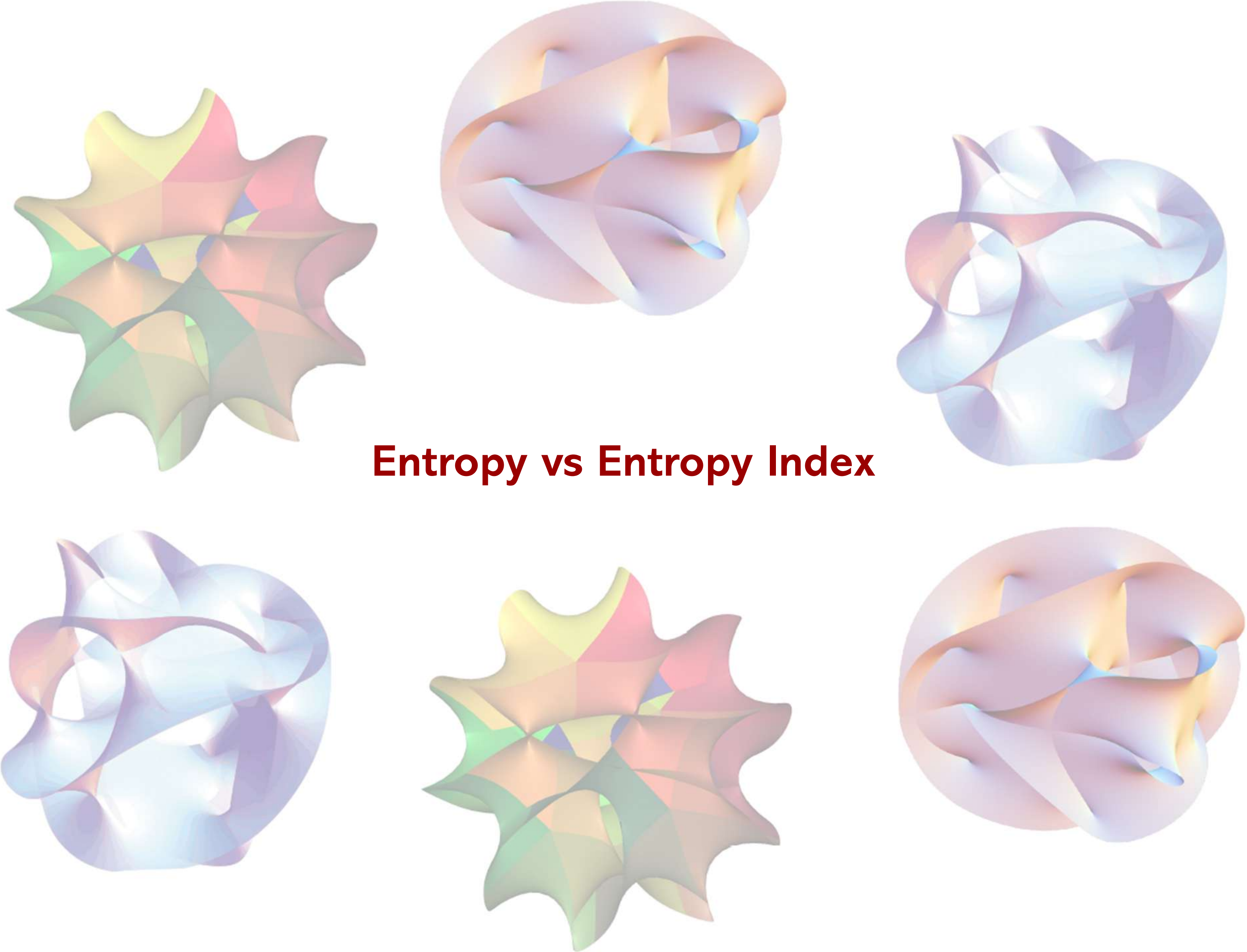
Thank you for your attention!

Contact: acastellano@uchicago.edu

Back-up Slides



Entropy vs Entropy Index



What do we mean by Entropy?

- The previous formula arises by using Wald's formalism in a truncated theory [Lopes-Cardoso, Wit, Mohaupt '99]
- Essentially, one ignores D-term-like and hypermultiplet contributions
- Some of these were shown to give vanishing corrections [Lopes-Cardoso et al. '00; Murthy, Reys '13]
- It is believed that what we actually compute is a grav. index [Ooguri, Vafa, Strominger '04]

$$\mathcal{S}_{\rm BH} = \log \mathcal{Z}_{\rm index} - iq\phi \qquad \text{with} \quad \mathcal{Z}_{\rm index} = \mathrm{Tr}\left[(-1)^F e^{iq\phi}\right]_{\rm susy} = \sum_q (-1)^F \Omega(p,q)\, e^{iq\phi}$$

- In the large charge expansion one would have

$$\mathcal{Z}_{\rm index} \sim \mathcal{Z} \qquad \Longrightarrow \mathcal{S}_{\rm micro} = \log \Omega(p,q) \sim \mathcal{S}_{\rm BH}$$

[Zaffaroni '19]

What do we mean by Entropy?

- The previous formula arises by using Wald's formalism in a truncated theory [Lopes-Cardoso, Wit, Mohaupt '99]
- Essentially, one ignores D-term-like and hypermultiplet contributions
- Some of these were shown to give vanishing corrections [Lopes-Cardoso et al. '00; Murthy, Reys '13]
- It is believed that what we actually compute is a grav. index [Ooguri, Vafa, Strominger '04]

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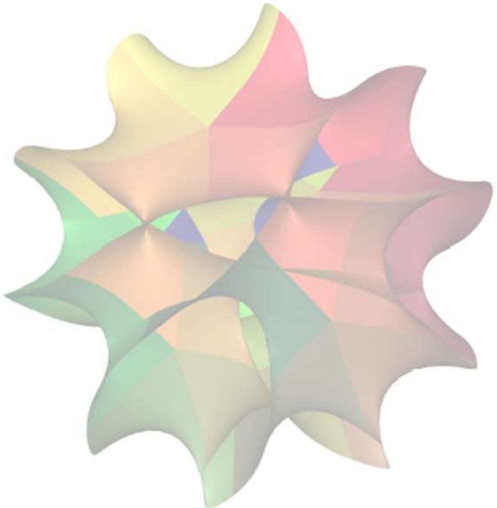
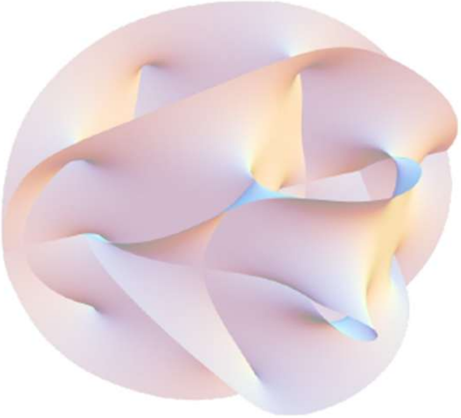
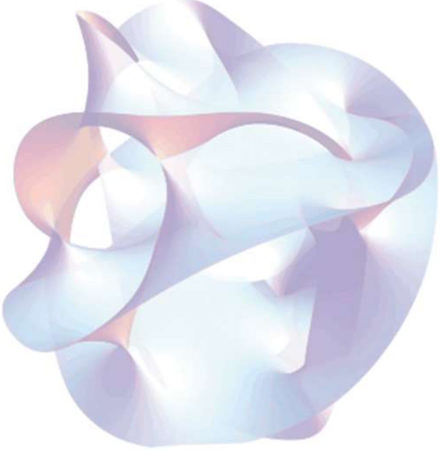
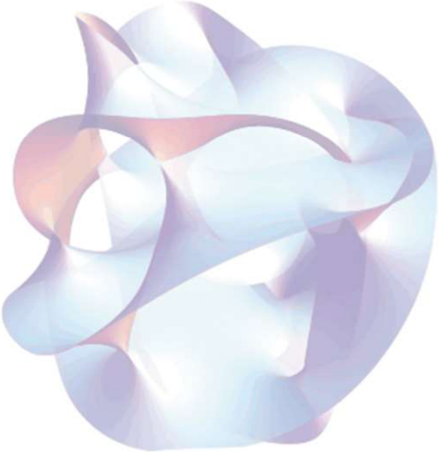
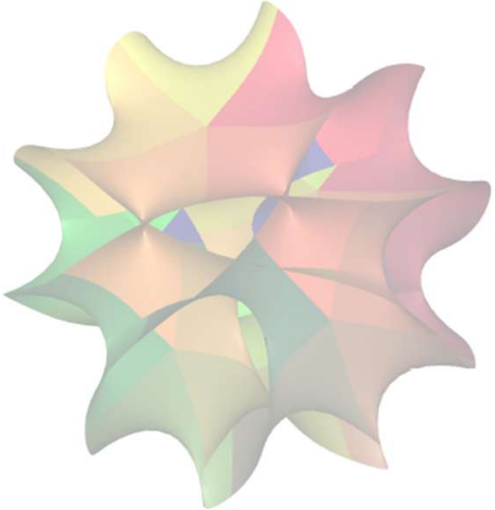


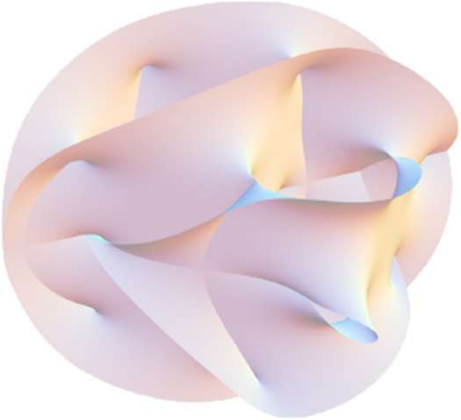




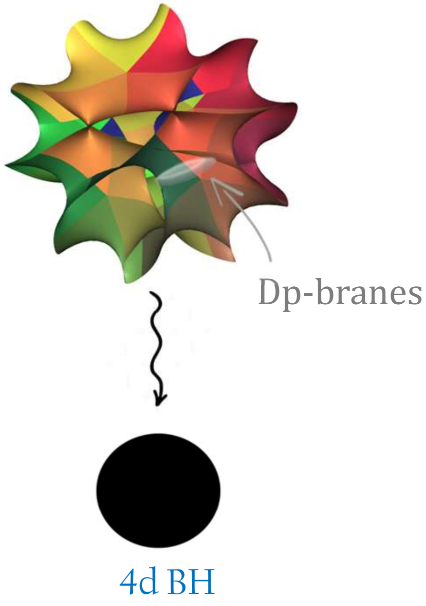
D0-D2-D4 BH System







The D0-D2-D4 BH System



- Consider BPS BHs with no D6-brane charge

- The two-derivative attractor solution is well known. We thus impose

$$W^2 \to 0\,, \quad F(X^A, W^2) \to \mathcal{F}(X^A)$$

- The solution reads [Shmakova '96]

$$CX^a = \frac{1}{6} CX^0 D^{ab} q_b + \frac{i}{2} p^a \qquad (CX^0)^2 = \frac{1}{4} \frac{D_{abc} p^a p^b p^c}{\hat{q}_0} \equiv (x^0)^2$$

- From here one may easily determine both the radius and the entropy of the BH system

$$\frac{r_h^2}{G_4} = |Z(q_A, p^B)|^2 = -\frac{D_{abc} p^a p^b p^c}{CX^0} = 2\sqrt{\frac{1}{6} |\hat{q}_0| \mathcal{K}_{abc} p^a p^b p^c}$$

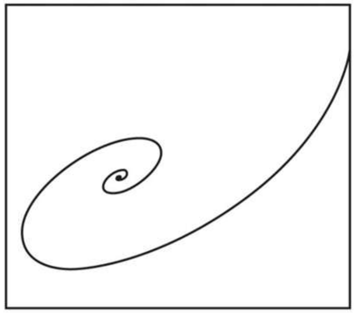
$$\mathcal{S}_{\rm BH}(q_A, p^B) = -4\pi CX^0 \hat{q}_0 = 2\pi \sqrt{\frac{1}{6} |\hat{q}_0| \left(\mathcal{K}_{abc} p^a p^b p^c\right)}$$

The D0-D2-D4 BH System

- We assumed large vol approximation $\rightsquigarrow$ need to ensure that the solution is consistent!
- Due to monotonicity of BPS flow, we only have to worry about the horizon locus [Ferrara '95-'97]
- Compute stabilized volumes:

$$t^a_{\rm h} = {\rm Im}\left(\frac{CX^a}{CX^0}\right)\bigg|_{\rm hor} = p^a\sqrt{\frac{6|\hat{q}_0|}{\mathcal{K}_{abc}p^ap^bp^c}}$$

$$\mathcal{V}_{\rm h} = \frac{1}{8}e^{-K}\,|X^0|^{-2}\bigg|_{\rm hor} = \frac{1}{8}\frac{|Z|^2}{|CX^0|^2} = \sqrt{\frac{6|\hat{q}_0|^3}{\mathcal{K}_{abc}p^ap^bp^c}}$$



- Thus we need to impose the following charge hierarchy

$$(\hat{q}^0)^2,\ (p^a)^2 \gg \left|\frac{D_{abc}p^ap^bp^c}{\hat{q}_0}\right| \implies |\hat{q}^0| \gg p^a$$

$\rightsquigarrow$ We do not specify x_0

Including Perturbative Quantum Corrections

- Taking now the generalized prepotential with the leading quant. corrections yields

$$(Y^0)^2 = \frac{\frac{1}{4}D_{abc}p^a p^b p^c - d_a p^a \Upsilon}{\hat{q}_0 + i(G_0 - \bar{G}_0)}\,, \qquad \text{with} \quad G_0 \equiv \frac{\partial G(Y^0, \Upsilon)}{\partial Y^0}$$

[Lopes-Cardoso, Wit, Mohaupt '99]

$$Y^a = \frac{1}{6} Y^0 D^{ab} q_b + \frac{i}{2} p^a$$

↜ *Same as before!*

- Notice that in order to recover the previous classical solution we need to impose

$$|\hat{q}_0| \gg p^a \gg 1\,, \qquad |\hat{q}_0| \gg |i(G_0 - \bar{G}_0)|$$

- One can thus find an iterative solution of the form

$$(Y^0)^2 = (y^0)^2 \left(1 + \frac{i(G_0(y^0, \Upsilon) - \bar{G}_0(\bar{y}^0, \bar{\Upsilon}))}{|\hat{q}_0|} + \dots\right) \qquad \text{with} \quad (y^0)^2 = (x^0)^2 \left(1 - 4 d_a p^a \Upsilon / D_{bce} p^b p^c p^e\right)$$

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Including Perturbative Quantum Corrections

- The corrected black hole entropy and radius read as [Lopes-Cardoso, Wit, Mohaupt '99]

$$|\mathcal{Z}|^2 = 2\sqrt{\frac{1}{6}|\hat{q}_0|\mathcal{K}_{abc}p^a p^b p^c} + \dots$$

$$\mathcal{S}_{\rm BH} = 2\pi\sqrt{\frac{1}{6}|\hat{q}_0|\left(\mathcal{K}_{abc}p^a p^b p^c + c_{2,a}\,p^a\right)} - 2\pi i\left(G(y^0,\Upsilon) - \bar{G}(\bar{y}^0,\bar{\Upsilon})\right) + \dots$$

- We can understand the condition $|\hat{q}_0| \gg |i(G_0 - \bar{G}_0)|$ by evaluating the series

$$\begin{aligned} i\left(\bar{G}_0 - G_0\right) &= -\frac{\chi_E(X_3)}{(2\pi)^3}\,Y^0 \sum_{g=0,2,3,\dots} (2-2g)\,c^3_{g-1}\,\alpha^{2g} + \dots \\ &= -\frac{\chi_E(X_3)}{8(2\pi)^3}\,|\Upsilon|^{1/2} \sum_{g=0,2,3,\dots} (2-2g)\,c^3_{g-1}\,\alpha^{2g-1} + \dots \end{aligned}$$

- We are forced to refine the previous charge hierarchy as follows

$$(\hat{q}^0)^2,\,(p^a)^2 \gg \left|\frac{D_{abc}p^a p^b p^c}{\hat{q}_0}\right| \gg 1\,, \qquad \text{with} \quad |\hat{q}^0|\,,\,p^a \gg 1$$

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$$\boxed{(\hat{q}^0)^2,\ (p^a)^2 \gg \left|\frac{D_{abc}p^a p^b p^c}{\hat{q}_0}\right| \gg 1\,, \qquad \text{with} \quad |\hat{q}^0|\,,\ p^a \gg 1}$$

Now we require large values for x_0

The Transition Regime

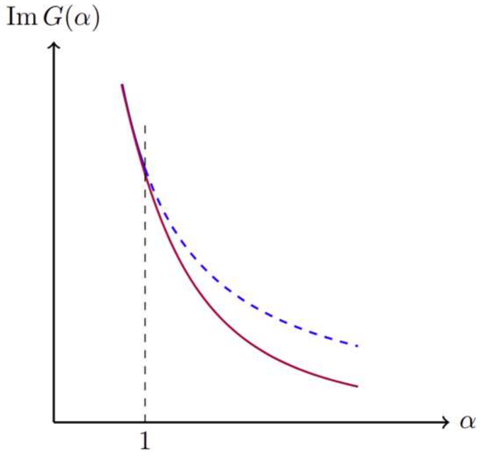
- Including leading quant. corrections yields sensible answers for certain charge hierarchies
- The latter are controlled by a series expansion that is asymptotic (for $\alpha \ll 1$)

$$G(Y^0, \Upsilon) \sim -\frac{i}{2(2\pi)^3}\, \chi_E(X_3)\, (Y^0)^2 \sum_{k=0}^{\infty} c^3_{k-1}\, \alpha^{2k} \qquad \text{with} \quad c^3_{k-1} \sim (2k-3)!$$

- The optimal truncation can be determined to behave as $N_\star \sim \frac{1}{2}\left(1 + \frac{4\pi^2}{\alpha}\right)$
- Thus, the series is invalidated for $\alpha \gtrsim \mathcal{O}(1)$ $\rightsquigarrow$ Interpretation?
- The attractor eqs actually tell us the physical meaning of

$$\boxed{|\alpha| = \frac{1}{8}\frac{|\Upsilon|^{1/2}}{|X^0|e^{\mathcal{K}/2}|\mathcal{Z}|} = \frac{\sqrt{8\mathcal{V}_{\rm h}}}{|\mathcal{Z}|} = \frac{r_5}{r_h}}$$

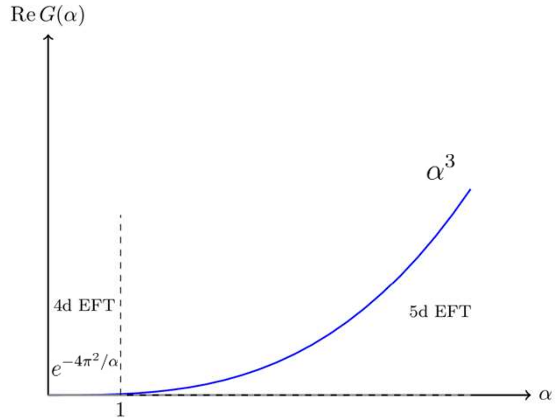
$$\text{since} \quad M_{\rm D0} = \sqrt{8\pi}|X^0|e^{\mathcal{K}/2}/\kappa_4\,,\, r_h = |\mathcal{Z}|\kappa_4/\sqrt{8\pi}$$



Including Non-Perturbative Effects

- The non-perturbative correction is now easily determined

$$\begin{aligned}\mathcal{I}^{(np)}(\alpha) &= -2\pi i\alpha \sum_{n,k=1}^{\infty} \frac{n}{k}\, e^{-\frac{4\pi^2 kn}{\alpha}} \left(1+\frac{\alpha}{4\pi^2 kn}\right) \\ &= -2\pi i\alpha \sum_{n=1}^{\infty}\left(n\,\mathrm{Li}_1\left(e^{-\frac{4\pi^2 n}{\alpha}}\right) + \frac{\alpha}{4\pi^2}\,\mathrm{Li}_2\left(e^{-\frac{4\pi^2 n}{\alpha}}\right)\right)\end{aligned}$$



- Notice the problematic growth for $\alpha \gg 1$

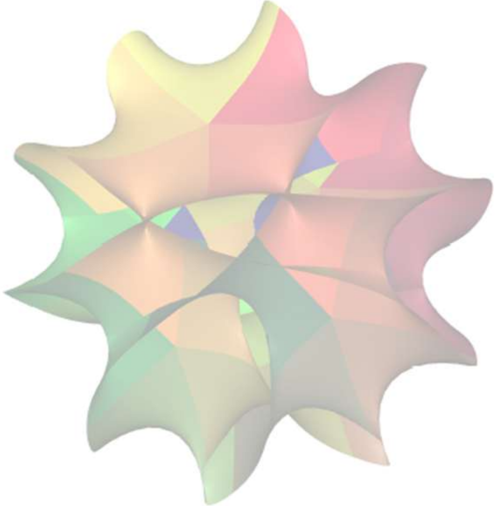
- Crucially, this has a different complex phase, and it does not enter the att. eqs nor BH obvs!

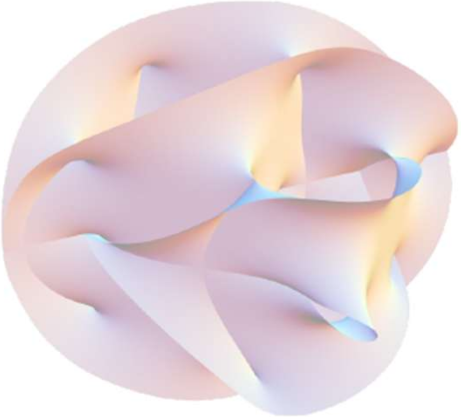
$$(Y^0)^2 = \frac{\frac{1}{4}D_{abc}p^ap^bp^c - d_ap^a\Upsilon}{\hat{q}_0 + i(G_0-\bar{G}_0)}$$

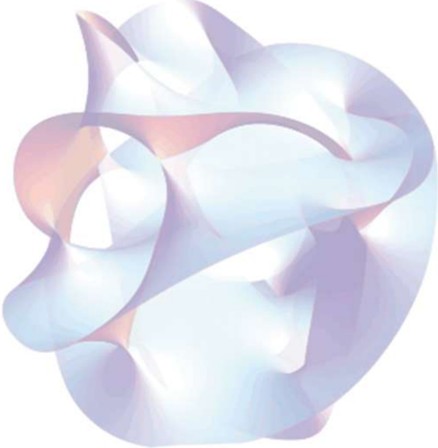
Attractor eq.

$$\begin{aligned}|\mathcal{Z}|^2 &= -\frac{D_{abc}p^ap^bp^c - 2d_ap^a\Upsilon}{Y^0} + iY^0\left(G_0-\bar{G}_0\right) \\ \mathcal{S}_{\mathrm{BH}} &= -4\pi Y^0\hat{q}_0 - i\pi\left(3Y^0G_0 + 2\Upsilon G_\Upsilon - \mathrm{h.c.}\right)\end{aligned}$$

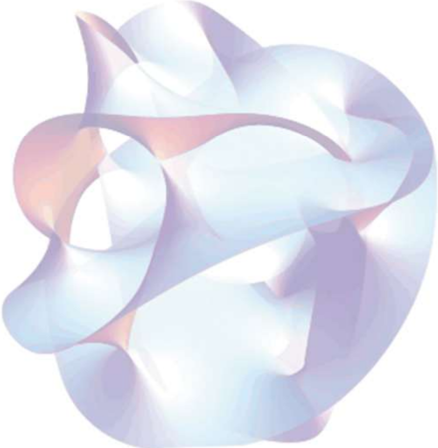
BH entropy and radius

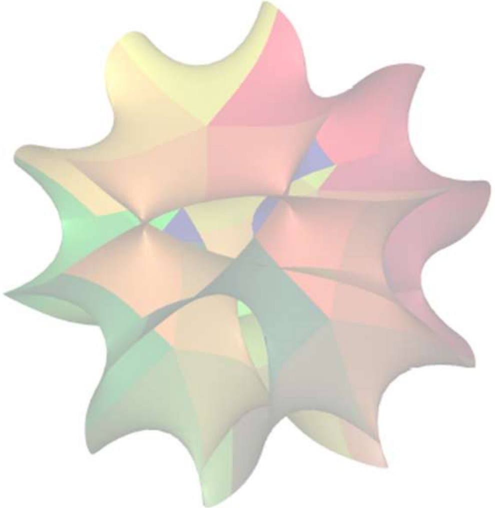


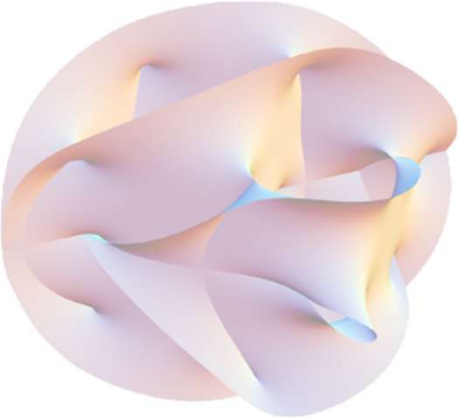




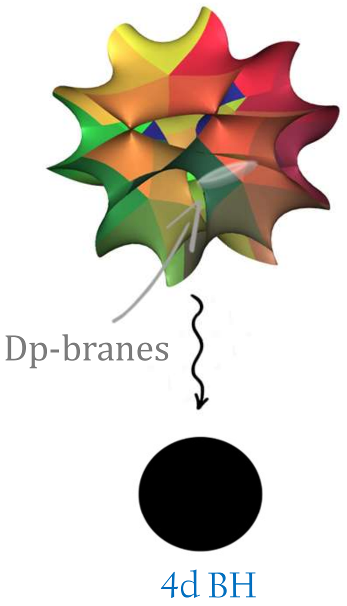
D2-D6 BH System







The (Classical) D2-D6 BH System



- We would now like to study other BPS solutions which include D6-brane charge
- At 2-derivatives the problem is hard: we must deal with a quadratic alg. system
- We focus on a particularly simple system, i.e. the D2-D6 BPS black hole
- Having no D4 charge implies

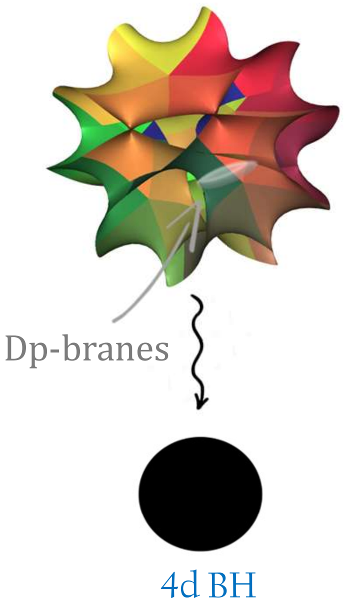
$$CX^0 = \mathrm{Re}\, CX^0 + i\,\frac{p^0}{2} \qquad CX^a = \bar{C}\bar{X}^a = \mathrm{Re}\, CX^a$$

- The attractor equations read

$$D_{abc}\,(CX^b)(CX^c) = -\frac{q_a}{3p^0}|CX^0|^2$$

$$q_0 = \frac{2\,p^0\,\mathrm{Re}\,CX^0\,\left(D_{abc}\,(CX^a)(CX^b)(CX^c)\right)}{|CX^0|^4} = -\frac{2\ \mathrm{Re}\,CX^0\,(q_a CX^a)}{3|CX^0|^2}$$

The (Classical) D2-D6 BH System



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- The attractor equations (imposing no D0 charge) read

$$D_{abc}\,(CX^b)(CX^c) = -\frac{q_a}{3p^0}|CX^0|^2$$
$$q_0 \propto CX^0 = 0$$

The (Classical) D2-D6 BH System

- Defining the variables [Shmakova '96]

$$x^a = \text{Re}\, CX^a \sqrt{\frac{3}{|CX^0|^2}} \Longrightarrow D_{abc}x^b x^c = -\frac{q_a}{p^0}$$

- One can easily write the physical properties of the BHs such as the central charge

$$|Z|^2 = -\frac{4}{3}(q_a CX^a)$$

- ...as well as the relevant volumes (implying the hierarchy $q_a \gg p^0$)

$$\mathcal{V}_{\text{h}} = \frac{1}{8}\frac{|Z|^2}{|CX^0|^2} = \frac{2}{3}\frac{(-q_a CX^a)}{(p^0)^2} = \frac{D_{abc}CX^a CX^b CX^c}{i\,(CX^0)^3} \qquad t^a_{\text{h}} = \text{Im}\left(\frac{CX^a}{CX^0}\right)\Big|_{\text{hor}} = -2\frac{CX^a}{p^0} = -\frac{1}{2}\frac{p^0 CX^a}{|CX^0|^2}$$

- ...and the classical (i.e. Bekenstein-Hawking) entropy

$$\mathcal{S}_{\text{BH}} = -\pi\frac{4}{3}(q_a CX^a)$$

The (Quantum) D2-D6 BH System

- The quantum corrected attractor solution reads

$$3D_{abc}Y^bY^c = -\frac{q_a}{p^0}|Y^0|^2 - d_a\Upsilon$$
$$0 = \frac{2\,p^0\,\mathrm{Re}\,Y^0\left(D_{abc}Y^aY^bY^c + d_aY^a\Upsilon\right)}{|Y^0|^4} - i\left(G_0 - \bar{G}_0\right)$$

- The formal series of corrections is now alternating

$$G(Y^0,\Upsilon) = \frac{i}{2(2\pi)^3}\,\chi_E(X_3)\,|Y^0|^2 \sum_{g=0,2,3,\dots} (-1)^g c^3_{g-1}\,|\alpha|^{2g}$$
$$\frac{\partial G(Y^0,\Upsilon)}{\partial Y^0} = \frac{\chi_E(X_3)}{2(2\pi)^3}|Y^0| \sum_{g=0,2,3,\dots} (-1)^g(2-2g)c^3_{g-1}\,|\alpha|^{2g}$$

G_0 is purely real!

- The solution is not spoiled! The relevant BH quantities are given by

$$|\mathcal{Z}|^2 = -\frac{4}{3}Y^a\left(q_a - \frac{1}{12p^0}c_{2\,a}\right) + p^0\,\mathrm{Re}\,G_0$$
$$\mathcal{S}_{\mathrm{BH}} = -\frac{4}{3}\pi Y^a\left(q_a + \frac{1}{6p^0}c_{2\,a}\right) + \frac{\chi_E(X_3)(p^0)^2}{4(2\pi)^2}\sum_{g=0,2,3,\dots}(-1)^g(2-2g)c^3_{g-1}\,|\alpha|^{2g}$$

No Genuine 5d Regime

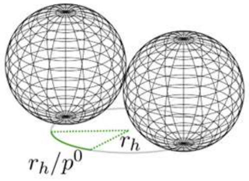
- Interestingly, $\alpha = i|\alpha|$ is upper bounded due to charge quantization

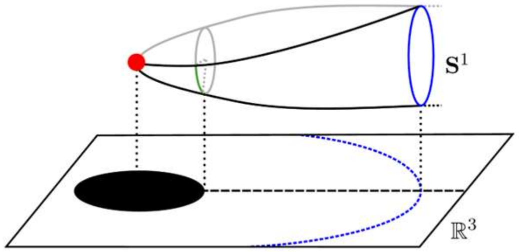
$$\alpha = -i\,|\alpha| \qquad |\alpha| = \frac{2}{p^0}$$

- From M-theory this is easily understood geometrically

- The BH can be understood as a 5d BH at the center of a Taub-NUT

- Still, one may explore the $r_h \gtrsim r_5$ regime





- The quantum series diverges, and we need a 5d regularization

Non Local and Non Perturbative Effects

- We resort again to the GV prescription, using that now $\alpha = i|\alpha|$

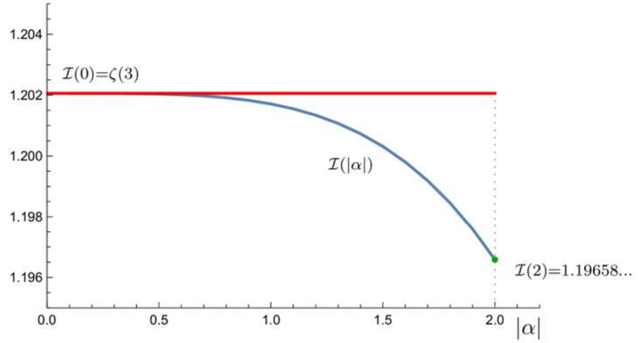
$$G(Y^0, \Upsilon) = -\frac{i}{2(2\pi)^3}\,\chi_E(X_3)\,|Y^0|^2\,\mathcal{I}(|\alpha|)$$

$$\mathcal{I}(|\alpha|) = \frac{|\alpha|^2}{4}\sum_{n\in\mathbb{Z}}\int_{0^+}^{\infty}\frac{\mathrm{d}s}{s}\,\frac{1}{\sin^2(\pi n|\alpha|s)}\,e^{-4\pi^2 n^2 i s}$$

Grav. analogue of
magnetic self-dual bckgrd

- Now we can freely deform the contour of integration without picking poles!
- The resulting integral can be performed numerically

$$\boxed{\mathcal{I}(|\alpha|) = \zeta(3) - \frac{|\alpha|^2}{2}\int_0^{\infty}\frac{\mathrm{d}s}{s}\,\frac{e^{-\frac{4\pi s}{|\alpha|}}}{1-e^{-\frac{4\pi s}{|\alpha|}}}\left(\frac{1}{\sinh^2(s)} - \frac{1}{s^2} + \frac{1}{3}\right)}$$

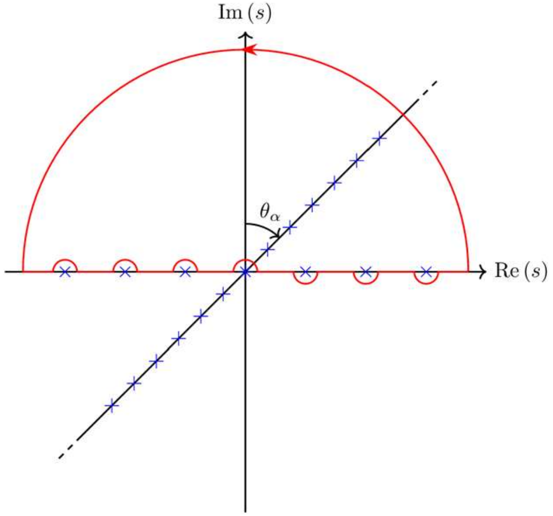


- The entropy is finite in the transition regime and there are no non pert. effects

Challenges with the Cauchy Formulation

- Question: Can we always use the simple Cauchy formula? [see also Hattab, Palti '24]

$$\mathcal{I}(\alpha) = \frac{\alpha^2}{4} \oint \frac{\mathrm{d}s}{s} \frac{1}{1-e^{-2\pi i s}} \frac{1}{\sinh^2\left(\frac{\alpha s}{2}\right)} \qquad \alpha = |\alpha| e^{i\theta_\alpha} \in \mathbb{C}$$



- The non perturbative poles are now rotated. They arise at

$$s = \frac{2\pi n}{|\alpha|} \exp\left(i\pi/2 - i\theta_\alpha\right), \qquad n \in \mathbb{Z}$$

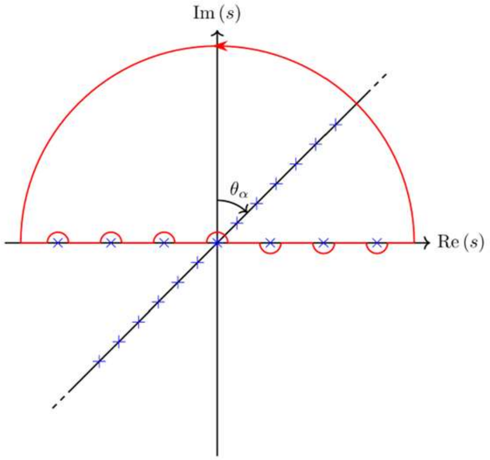
- The series of residues behave as

$$\mathcal{I}^{(p)}(\alpha) \sim \alpha^2 \sum_{k=1}^{\infty} \frac{1}{k} e^{-k\alpha}$$

$$\mathcal{I}^{(np)}(\alpha) \sim -2\pi i \alpha \sum_{n=1}^{\infty} \frac{1}{n} e^{-\frac{4n\pi^2}{\alpha}}$$

- Whenever $\alpha = i|\alpha|$ both sets of poles appear along the real axis!

Challenges with the Cauchy Formulation



- In that case the asymptotics changes dramatically

$$\mathcal{I}^{(p)}(\alpha) \sim -\alpha^2 \sum_{k=1}^{\infty} \frac{1}{4k \sin^2\left(\frac{k|\alpha|}{2}\right)}$$

$$\mathcal{I}^{(np)}(\alpha) \sim 2\pi i\alpha \sum_{n=1}^{\infty} \frac{1}{4n \sin^2\left(\frac{2n\pi^2}{|\alpha|}\right)}$$

$$\alpha = i|\alpha|$$

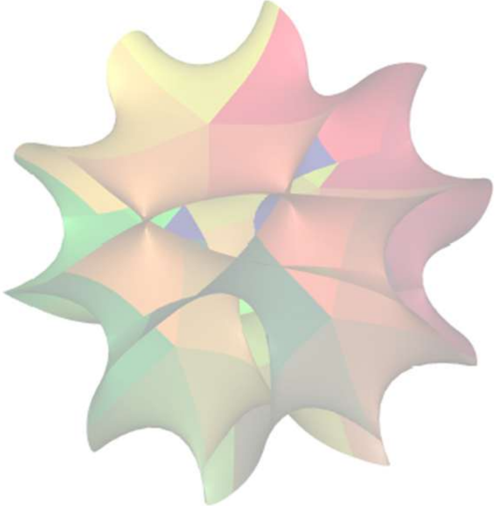
- They badly diverge for two reasons:
  1. The series are lower bounded by the harmonic one
  2. The series of residues are dominated by quasi-poles [Apostol '12: Dirichlet's approx. theorem]

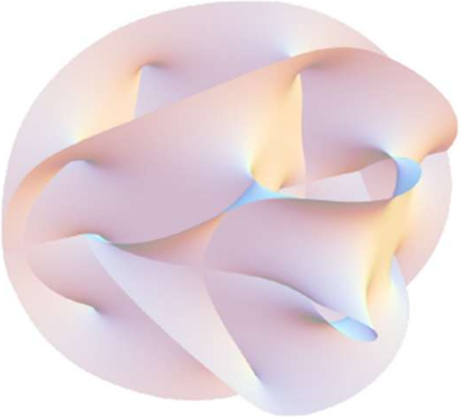
- There are infinitely many integer pairs satisfying

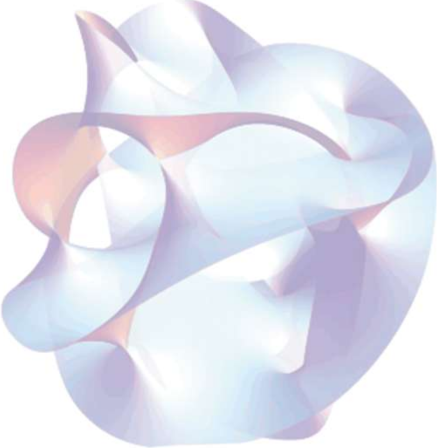
$$0 < \left|\gamma - \frac{p_\gamma}{q_\gamma}\right| < \frac{1}{q_\gamma^2} \Longrightarrow \frac{1}{\sin^2(p_\pi)} \sim \frac{1}{|\pi q_\pi - p_\pi|^2} > q_\pi^2 \sim \frac{p_\pi^2}{\pi^2}$$

- One cannot simply add the arc at infinity, but rather integrate over the imaginary axis

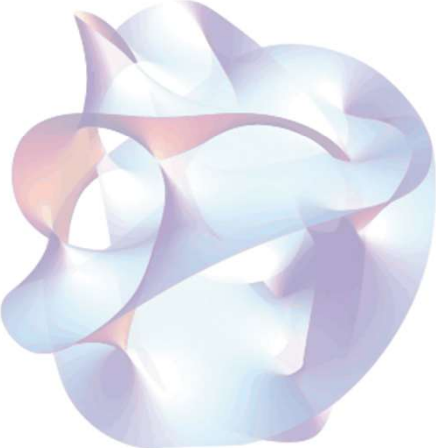
$$\boxed{\mathcal{I}(|\alpha|) = -\frac{|\alpha|^2}{4} \int_{-\infty}^{\infty} \frac{\mathrm{d}\tau}{\tau} \frac{1}{1 - e^{-2\pi\tau}} \frac{1}{\sinh^2\left(\frac{|\alpha|\tau}{2}\right)}}$$

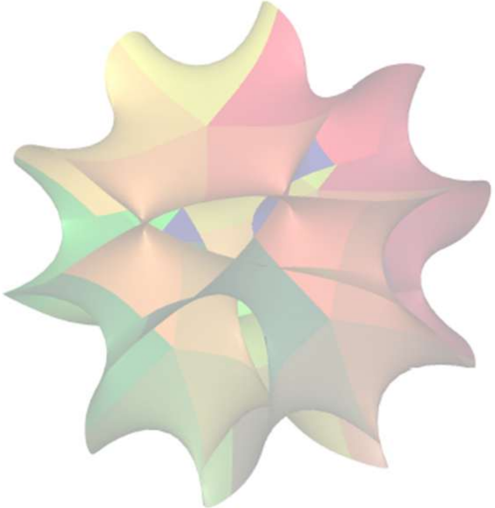


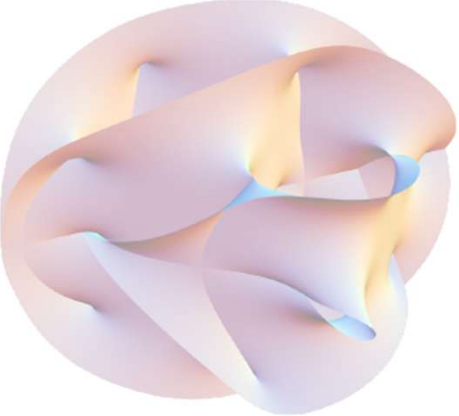




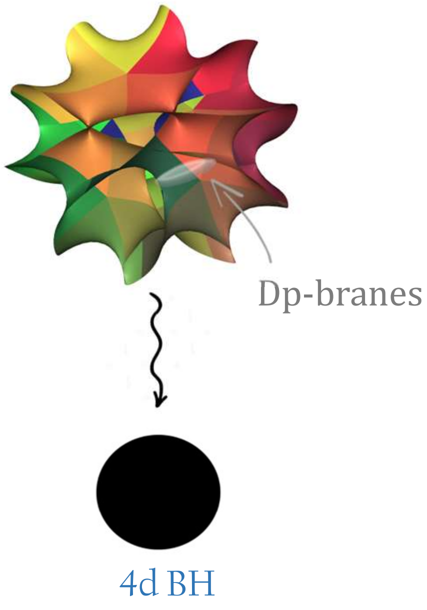
General BPS BH System







The General (3-level) BH Entropy Formula



- Consider BPS BHs with D0-D2-D4-D6 charges

- The two-derivative attractor solution is well known. We thus impose

$$W^2 \to 0\,, \qquad F(X^A, W^2) \to \mathcal{F}(X^A)$$

- The attractor eqs. lead to a quadratic system, in general

$$ip^A = e^{\mathcal{K}}\left(\mathcal{Z}X^A - \bar{\mathcal{Z}}\bar{X}^A\right) \qquad iq_A = e^{\mathcal{K}}\left(\mathcal{Z}\mathcal{F}_A - \bar{\mathcal{Z}}\bar{\mathcal{F}}_A\right)$$

- After some tedious algebra one can find the Bekenstein-Hawking entropy [Shmakova '96]

$$\mathcal{S}_{\mathrm{BH}}\Big|_{\mathrm{class}} = \frac{\pi}{3p^0}\sqrt{\frac{4}{3}\left(\tilde{\Delta}_a\tilde{x}^a\right)^2 - 9\left(p^0p^Aq_A - 2D_{abc}p^ap^bp^c\right)^2}$$

$$\text{with} \quad \tilde{\Delta}_a = 3D_{abc}p^bp^c - p^0q_a \qquad \tilde{\Delta}_a = D_{abc}\tilde{x}^b\tilde{x}^c \quad \leftsquigarrow \quad \textit{quadratic system}$$

The General (Quantum) BH Entropy Formula

- Recall that we restrict ourselves to the Large Volume regime

$$F(Y,\Upsilon) = \frac{D_{abc}Y^aY^bY^c}{Y^0} + d_a\,\frac{Y^a}{Y^0}\,\Upsilon + G(Y^0,\Upsilon)$$

Tree level

g = 1 corrections

Higher-genus corrections

- The resulting quantum entropy with arbitrary D0-D2-D4-D6 charges reads as

[AC, Lust, Montella, Zatti ‘25]

$$\boxed{\begin{aligned}\mathcal{S}_{\rm BH} = {}& \frac{\pi}{3p^0}\sqrt{\frac{4}{3}\left(\tilde{\Delta}_a\tilde{x}^a\right)^2 - 9\left(p^0p^A\tilde{q}_A - 2D_{abc}p^ap^bp^c\right)^2} \\ & + 4\pi\left[\mathrm{Im}\,G - \mathrm{Re}\,Y^0\mathrm{Im}\,G_0 - \frac{1}{\sqrt{3}}\frac{(\mathrm{Re}\,Y^0)^2}{|Y^0|^3}\tilde{x}^a d_a\Upsilon\right]\end{aligned}}$$

where $\left(p^0p^A\tilde{q}_A - 2D_{abc}p^ap^bp^c\right)|Y^0| = \dfrac{2}{3\sqrt{3}}\mathrm{Re}\,Y^0\tilde{x}^a\tilde{\Delta}_a$ and $x^a = \tilde{x}^a\,\dfrac{|Y^0|}{\sqrt{3}p^0}\,,\quad \tilde{\Delta}_a = D_{abc}\tilde{x}^b\tilde{x}^c = -\tilde{q}_ap^0 + 3D_{abc}p^bp^c$

Consistency Checks

- We can take some limits to see whether we recover known results

1) The classical D0-D2-D4-D6 solution [Shmakova '96]

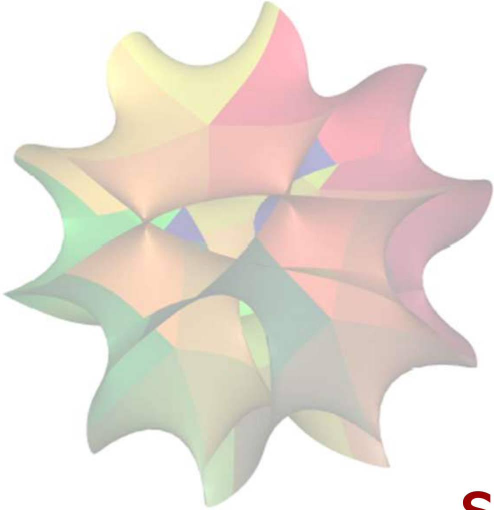
$$\mathcal{S}_{\rm BH}\big|_{\rm class} = \frac{\pi}{3p^0}\sqrt{\frac{4}{3}\left(\tilde{\Delta}_a \tilde{x}^a\right)^2 - 9\left(p^0 p^A q_A - 2D_{abc}p^a p^b p^c\right)^2} \quad \leftsquigarrow \quad \text{We set } d_a = G = 0$$

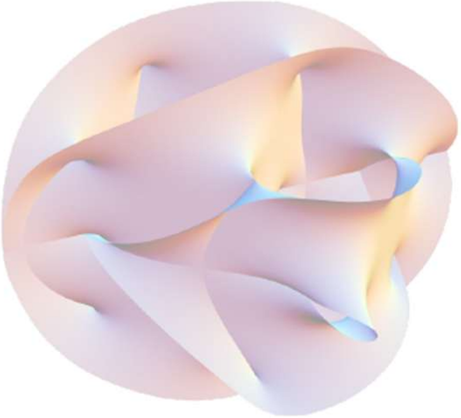
2) The quantum D2-D6 system [AC, Zatti '25]

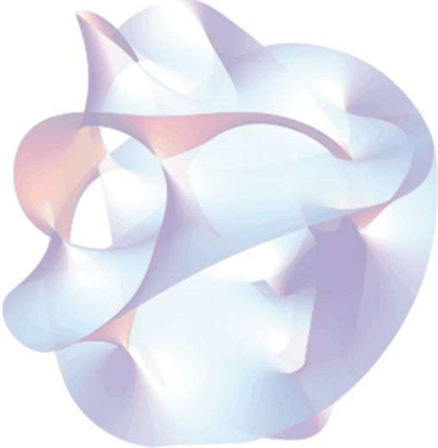
$$\mathcal{S}_{\rm BH}\big|_{\rm D2-D6} = -\frac{4\pi}{3}Y^a\left(q_a + \frac{4d_a\Upsilon}{p^0}\right) + 4\pi{\rm Im}\, G \quad \leftsquigarrow \quad \text{We set } p^a = q_0 = 0$$

3) The quantum D0-D2-D4 solution [Lopes-Cardoso, Wit, Mohaupt '99]

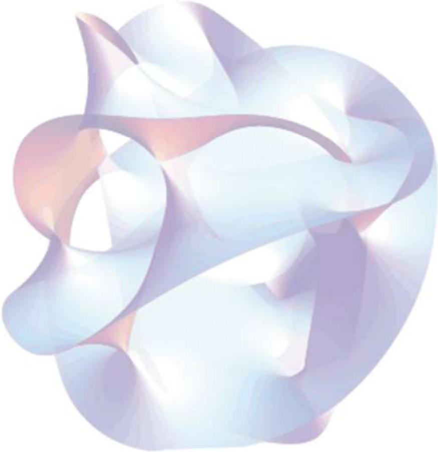
$$\begin{aligned}\mathcal{S}_{\rm BH}\big|_{\rm D0-D2-D4} &= -\pi\frac{D_{abc}p^a p^b p^c}{Y^0} + 4\pi{\rm Im}\, G - 4\pi Y^0{\rm Im}\, G_0 + 4\pi\frac{p^a d_a\Upsilon}{Y^0} \quad \leftsquigarrow \quad \text{We take the } p^0 \to 0 \text{ limit}\\ &= -4\pi\hat{q}_0 Y^0 + 4\pi{\rm Im}\, G + 4\pi Y^0{\rm Im}\, G_0\end{aligned}$$

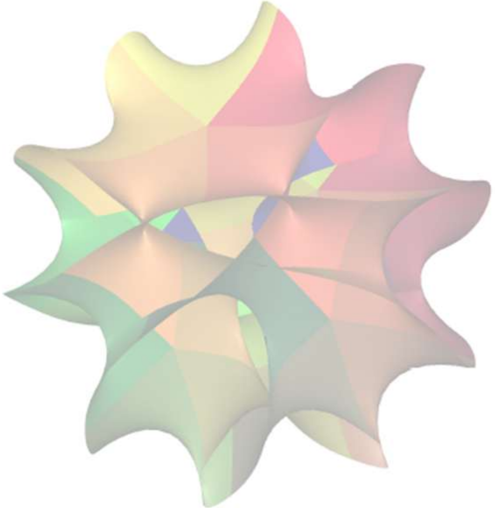


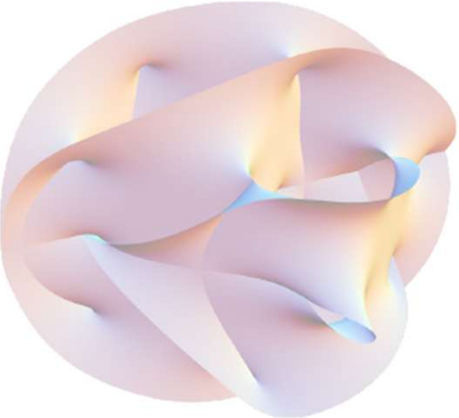




Symplectic Transformations







A Useful (Symplectic) Trick

- The result is nice… but the formulas are ugly and somewhat intractable
- We can simplify the problem using a $Sp(2n_V+2,\mathbb{Z})$ symmetry [de Wit, van Proeyen '84]

$$S \supset \frac{1}{4\kappa_4^2}\int F^{A,-}\wedge G_A^- + \text{h.c.} \qquad \text{with} \quad F_{\mu\nu}^{A,-} = \frac{1}{2}\left(F_{\mu\nu}^A + i\star F_{\mu\nu}^A\right)\,, G_A^- = \bar{\mathcal{N}}_{AB}F^{B,-}$$

EOM invariant under

$$\begin{pmatrix} F^{A,-} \\ G_A^- \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}\begin{pmatrix} F^{A,-} \\ G_A^- \end{pmatrix}$$

$$\mathcal{A}^{\mathrm{T}}\mathcal{D}-\mathcal{C}^{\mathrm{T}}\mathcal{B} = \mathbf{1}_{n_V+1}$$
$$\mathcal{A}^{\mathrm{T}}\mathcal{C}-\mathcal{C}^{\mathrm{T}}\mathcal{A} = \mathcal{B}^{\mathrm{T}}\mathcal{D}-\mathcal{D}^{\mathrm{T}}\mathcal{B} = 0$$

- At 2-derivatives this can be encapsulated by the prepotential transformation law [Cecotti et al '89]

$$\begin{aligned} F'(X^{A\prime}) \;&=\; F(X^A) - \frac{1}{2}X^A\mathcal{F}_A + \frac{1}{2}X^A(\mathcal{C}^{\mathrm{T}}\mathcal{B}+\mathcal{A}^{\mathrm{T}}\mathcal{D})_A{}^B\mathcal{F}_B \\ &+ \tfrac{1}{2}X^A(\mathcal{C}^{\mathrm{T}}\mathcal{A})_{AB}X^B + \tfrac{1}{2}\mathcal{F}_A(\mathcal{D}^{\mathrm{T}}\mathcal{B})^{AB}\mathcal{F}_B \end{aligned}$$

A Useful (Symplectic) Trick

- The previous story easily generalizes when F-terms are included! [de Wit '96]

$$F'(Y^{A\prime}) \;=\; F(Y^A) - \frac{1}{2}Y^A F_A + \frac{1}{2}Y^A(\mathcal{C}^{\mathrm{T}}\mathcal{B} + \mathcal{A}^{\mathrm{T}}\mathcal{D})_A^{\;B} F_B$$
$$+\, \tfrac{1}{2}Y^A(\mathcal{C}^{\mathrm{T}}\mathcal{A})_{AB}Y^B + \tfrac{1}{2}F_A(\mathcal{D}^{\mathrm{T}}\mathcal{B})^{AB}F_B$$

- The BH entropy is crucially a symplectic function [Lopes-Cardoso, Wit, Mohaupt '99]

$$\frac{\mathcal{S}_{\mathrm{BH}}}{\pi} = |\mathcal{Z}|^2 + 4\mathrm{Im}\,(\Upsilon\partial_\Upsilon F(Y,\Upsilon)) = \mathbf{q}^{\mathrm{T}}\cdot\eta\cdot C\mathbf{\Pi} \quad + \quad \mathrm{Im}(\Upsilon\partial_\Upsilon F) \;=\; \frac{\mathcal{S}'_{\mathrm{BH}}}{\pi}$$

Symplectic pairing

Symplectic invariant

- Therefore, by choosing wisely a symplectic map, we can relate two diff. BH systems

A Useful (Symplectic) Trick

- We choose the following transformation [Gaiotto, Strominger, Yin '05]

$$\begin{pmatrix} Y^{0\prime} \\ Y^{a\prime} \\ F_0' \\ F_a' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{\gamma^a}{p^0} & \delta^a_b & 0 & 0 \\ -D_{abc}\frac{\gamma^a\gamma^b\gamma^c}{(p^0)^3} & -3D_{bcd}\frac{\gamma^c\gamma^d}{(p^0)^2} & 1 & -\frac{\gamma^a}{p^0} \\ 3D_{abc}\frac{\gamma^b\gamma^c}{(p^0)^2} & 6D_{abc}\frac{\gamma^c}{(p^0)^2} & 0 & \delta^b_a \end{pmatrix} \begin{pmatrix} Y^0 \\ Y^b \\ F_0 \\ F_b \end{pmatrix}$$

Real coeffs

- The BH charges transform as follows

$$p^{0\prime} = p^0\,, \qquad p^{a\prime} = p^a + \gamma^a\,, \qquad q_a' = q_a + 6D_{abc}\frac{p^b\gamma^c}{p^0} + 3D_{abc}\frac{\gamma^b\gamma^c}{p^0}\,,$$

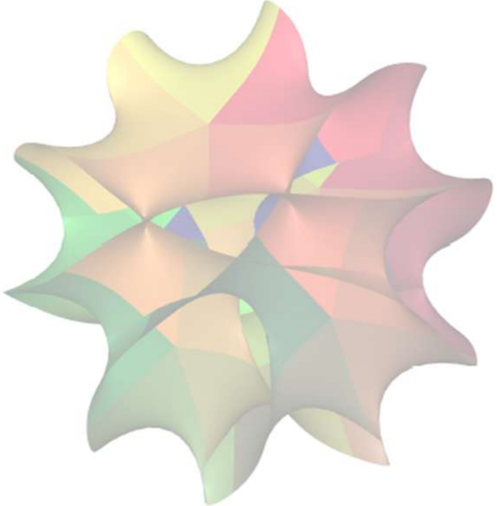
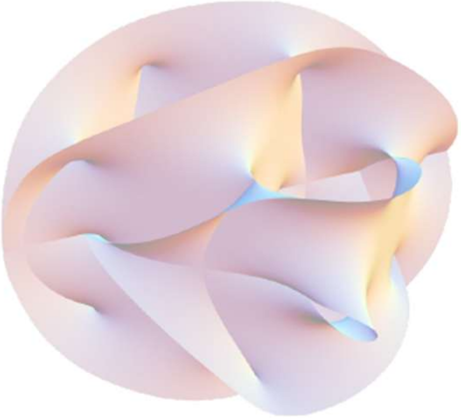
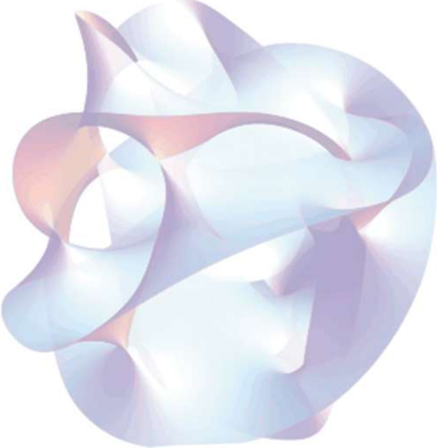
$$q_0' = q_0 - \frac{\gamma^a q_a}{p^0} - 3D_{abc}\frac{\gamma^a\gamma^b p^c}{(p^0)^2} - D_{abc}\frac{\gamma^a\gamma^b\gamma^c}{(p^0)^2}$$

- The functional form of the prepotential is left invariant (symplectic symmetry)

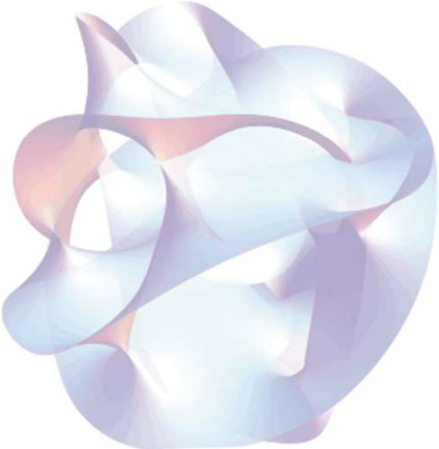
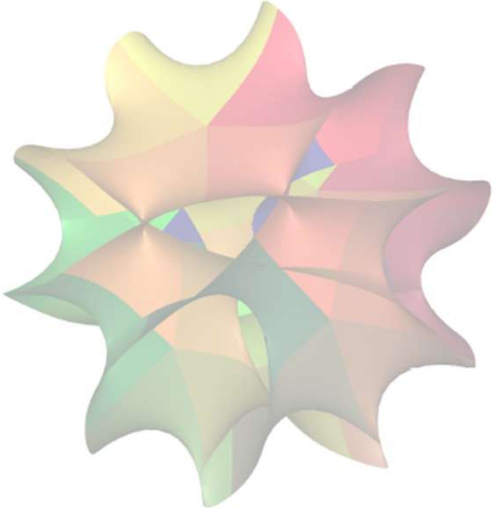
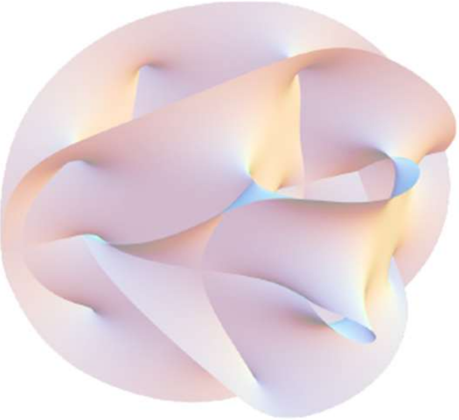
$$F'(Y^{A\prime},\Upsilon') = F(Y^A,\Upsilon) + 3\frac{\gamma^a D_{abc}Y^bY^c}{p^0} + 3\frac{\gamma^a\gamma^b D_{abc}Y^0Y^c}{(p^0)^2} + \frac{D_{abc}\gamma^a\gamma^b\gamma^c(Y^0)^2}{(p^0)^3}$$

$$= \frac{D_{abc}Y^{a\prime}Y^{b\prime}Y^{c\prime}}{Y^{0\prime}} + d_a\frac{Y^{a\prime}}{Y^{0\prime}}\Upsilon' + G(Y^{0\prime},\Upsilon') - d_a\frac{\gamma^a}{p^0}\Upsilon' = F(Y^{A\prime})$$

Entropy preserved!

Details on Geodesics

BPS Geodesic Motion in AdS$_2$

- The worldline action has a $SL(2,\mathbb{R}) \times SU(2)$ symmetry

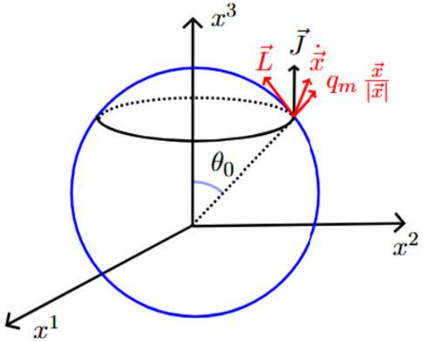
$$2H = \delta^{ik} J_i J_k + K_0^2 - \frac{1}{2}\{K_+, K_-\} - q_e^2 - q_m^2 = C_2^{\mathbf{S}^2} + C_2^{\mathrm{AdS}_2} - \tilde{m}^2$$

Motion is integrable

- The Hamiltonian constraint reduces to

$$H \overset{!}{=} -\frac{1}{2}\tilde{m}^2 \iff C_2^{\mathbf{S}^2} + C_2^{\mathrm{AdS}_2} = 0$$

Quadratic Casimirs

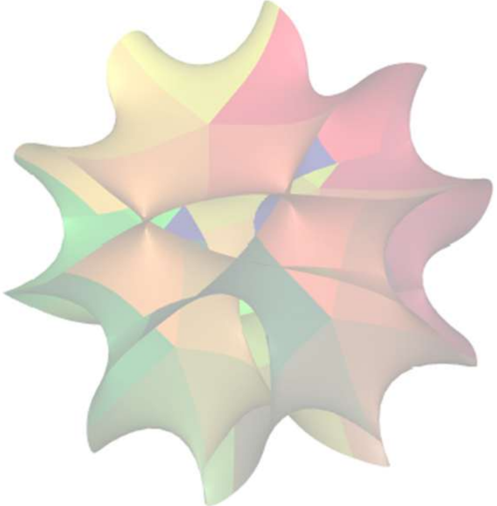


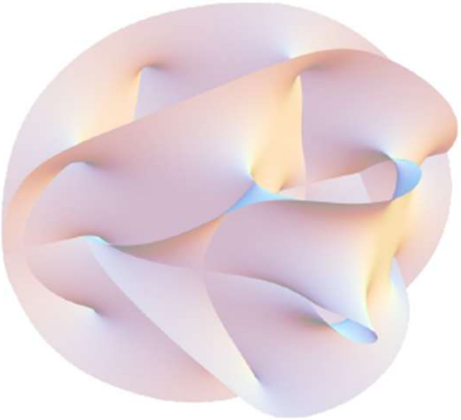
$$\cos\theta_0 = -\frac{q_m}{j}$$

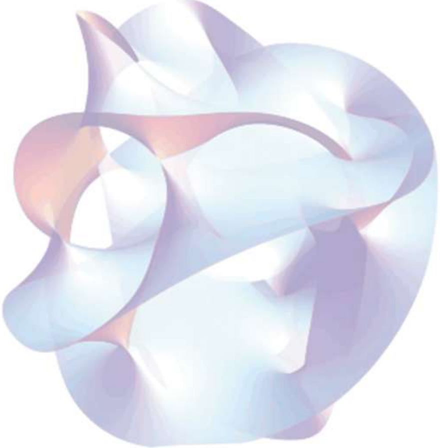
- The AdS trajectories depend on the relative size of electric charge and mass [AC, Lust, Montella, Zatti '25]

$$p_\rho^2 + V(\rho) = 0 \qquad \text{with} \quad V(\rho) = \frac{\tilde{m}^2 + \ell^2}{\rho^2} - \left(E + \frac{q_e}{\rho}\right)^2$$

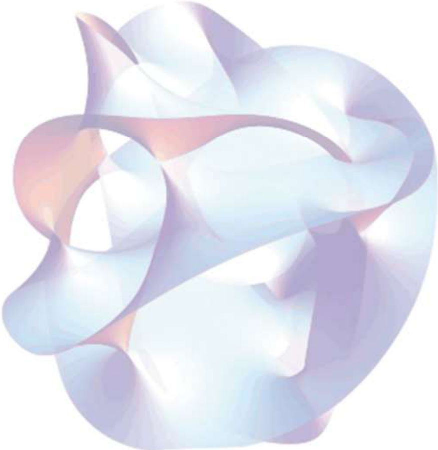
Charge-to-mass

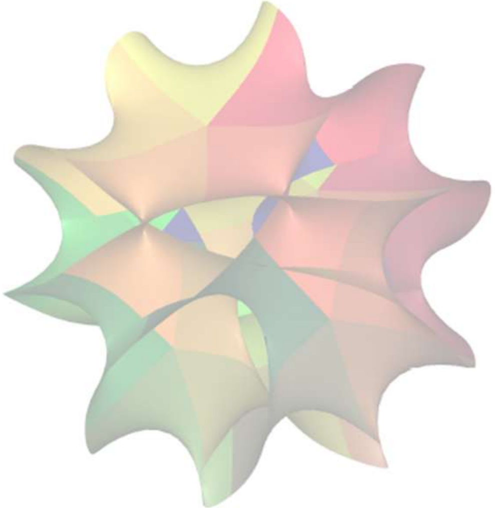


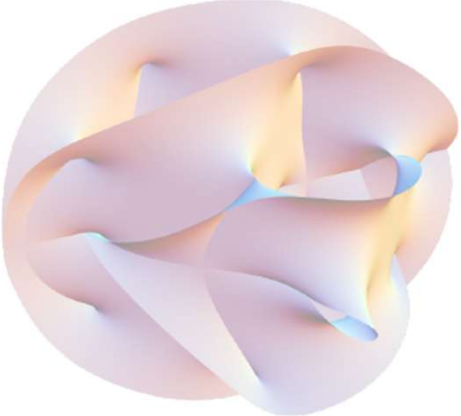




A Compendium of Landau Problems and Solutions







Warm-Up: Landau Problem in Flat Space

- Consider a charged scalar field in 2d Euclidean space within a constant mag. field

$$H_{\mathbb{R}^2} = -\left(\partial - iA\right)^2 \qquad A_i = -\frac{B}{2}\epsilon_{ij}x^j$$

- Using a complex coordinate $z = \sqrt{B/2}(x^1 + ix^2)$, the Landau Hamiltonian reads [Dunne '91]

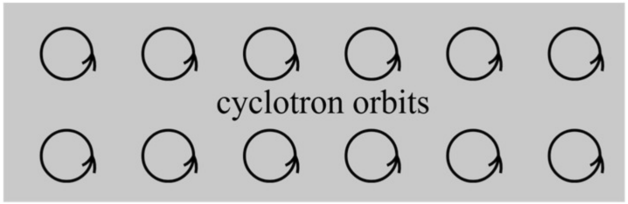
$$H_{\mathbb{R}^2} = 2B\left(-\partial\bar{\partial} - \frac{1}{2}(z\partial - \bar{z}\bar{\partial}) + \frac{1}{4}|z|^2\right)$$

- It clearly commutes with $J = z\partial - \bar{z}\bar{\partial}$, such that we can simultaneously diagonalize both

- It is convenient to rewrite this using the Heisenberg algebra (pair of commuting *oscillators*)

$$H_{\mathbb{R}^2} = 2B\left(a^\dagger a + \frac{1}{2}\right) \qquad J = b^\dagger b - a^\dagger a$$

$$\left[a, a^\dagger\right] = 1 \qquad \left[b, b^\dagger\right] = 1 \qquad \left[b, a^\dagger\right] = 0$$



Energy spectrum (Labdau levels)

$$\rightsquigarrow \left[\begin{array}{l} E_n = 2B\left(n + \frac{1}{2}\right) \qquad n \in \mathbb{Z}_{\geq 0} \\ j = -n, -n+1, \ldots \end{array}\right.$$

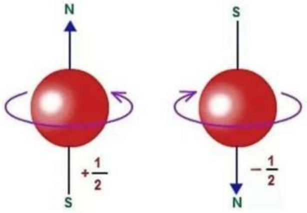
Warm-Up: Landau Problem in Flat Space

- The degeneracy in *J* leads to a constant density of eigenstates such that

$$\mathrm{Tr}\left(e^{sD^2_{\mathbb{R}^2}}\right) = \frac{1}{4\pi}\int d^2x\, \frac{B}{\sinh(Bs)}$$

- The spin ½ particle is similar. One needs to quantize the Dirac operator $\gamma^j D_j = -i\gamma^j(\nabla_j - iA_j)$

… or equivalently its square



$$\left(\gamma^j D_j\right)^2 = D^2 - B\sigma_z \qquad D^2 = \delta^{ij}\left(\partial_i - iA_i\right)\left(\partial_j - iA_j\right)$$

- This leads to a shifted spectrum between positive and negative chirality states

Energy spectrum
(Landau levels)

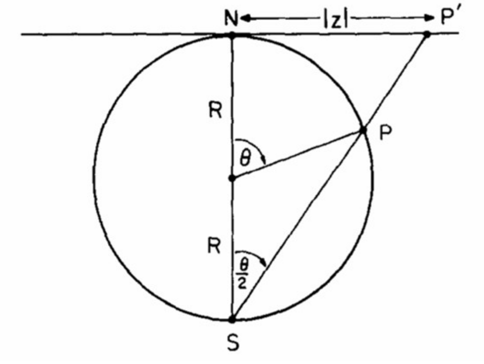
$$\rightsquigarrow \quad E_n^+ = 2Bn \quad E_n^- = 2B(n+1) \qquad n \in \mathbb{Z}_{\geq 0}$$

- The 1-loop determinant is determined by the trace of the Heat Kernel

$$\mathrm{Tr}\left[e^{s\left(\gamma^j D_j\right)^2}\right] = \frac{1}{2\pi}\int d^2x\, B\coth(Bs)$$

The Landau Problem in S^2

- Consider now charged scalar field in S^2 within a constant magnetic field [Dunne '91]



$$F = B\,\omega_{\mathbf{S}^2} = g\,\sin\theta\,d\theta\wedge d\phi \qquad \omega_{\mathbf{S}^2} = R^2\sin\theta\,d\theta\wedge d\phi$$

- The magnetic charge is quantized as $2g\in\mathbb{Z}$, and the Hamiltonian reads

$$H_{\mathbf{S}^2} = 2B\left(-\left(1+\frac{|z|^2}{2g}\right)^2\partial\bar\partial - \frac{1}{2}\left(1+\frac{|z|^2}{2g}\right)(z\partial-\bar z\bar\partial) + \frac{1}{4}|z|^2\right)$$

in stereographic coordinates $z=\sqrt{2g}\,e^{i\phi}\tan\left(\frac{\theta}{2}\right)$.

- To obtain the spectrum, we follow the same strategy and exploit the SU(2) symmetry

$$J_+ = -\frac{1}{\sqrt{2g}}z^2\partial - \sqrt{2g}\,\bar\partial + \sqrt{\frac{g}{2}}\,z \qquad J_- = \sqrt{2g}\,\partial + \frac{1}{\sqrt{2g}}\bar z^2\bar\partial + \sqrt{\frac{g}{2}}\,\bar z \qquad J_0 = z\partial - \bar z\bar\partial - g$$

which verify $[J_+,J_-]=2J_0\,,\quad [J_0,J_\pm]=\pm J_\pm$, such that $H_{\mathbf{S}^2}=B(C_2-g^2)/g$

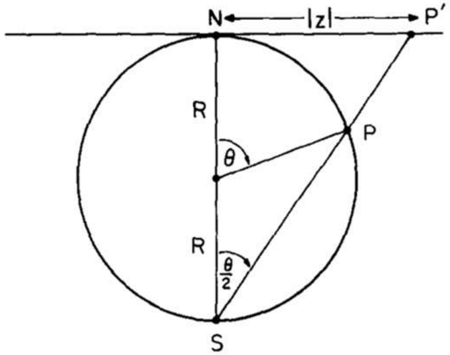
- This ultimately leads to a spectrum of the form: $\begin{cases} E_n = B\left(n+\frac{1}{2}+\frac{n(n+1)}{2g}\right) \\ \text{with}\quad d_n = 2j+1 = 2\left(g+n+\frac{1}{2}\right)\end{cases}$ [Wu, Yang '76]

The Landau Problem in S^2

- Next, consider a charged spin ½ field in S^2 within a constant magnetic field
- The Hamiltonian reads in stereographic coordinates $z = \sqrt{2g}\, e^{i\phi} \tan\left(\frac{\theta}{2}\right)$

$$\not{D}^2 = 2B\left[-\left(1+\frac{|z|^2}{2g}\right)^2 \partial\bar{\partial} - \frac{1}{2}\left(1+\frac{|z|^2}{2g}\right)(z\partial - \bar{z}\bar{\partial}) + \frac{1}{4}|z|^2\right]$$

$$+\frac{1}{R^2}\left[g\,\frac{\left(1+|z|^2/2g\right)^2}{8|z|^2} + \frac{1}{4} - g\sigma_z \frac{1-|z|^4/4g^2}{2|z|^2}(z\partial - \bar{z}\bar{\partial}) + \frac{g}{2}\,\left(1-|z|^2/2g\right)\sigma_z\right] - B\sigma_z$$



- To obtain the spectrum, we again exploit the SU(2) symmetry

$$J_+ = -\frac{1}{\sqrt{2g}} z^2\partial - \sqrt{2g}\,\bar{\partial} + \sqrt{\frac{g}{2}}\,z - \sqrt{\frac{g}{2}}\,\frac{1+|z|^2/2g}{\bar{z}}\frac{\sigma_z}{2} \qquad J_0 = z\partial - \bar{z}\bar{\partial} - g$$

$$J_- = \sqrt{2g}\,\partial + \frac{1}{\sqrt{2g}}\bar{z}^2\bar{\partial} + \sqrt{\frac{g}{2}}\,\bar{z} - \sqrt{\frac{g}{2}}\,\frac{1+|z|^2/2g}{z}\frac{\sigma_z}{2}$$

such that $C_2 = J_0^2 + \frac{1}{2}(J_+J_- + J_-J_+) = R^2\not{D}^2 + g^2 - \frac{1}{4}$.

- The spectrum includes 2g zero modes and (2x) the scalar one with $j = n + g - \frac{1}{2}$

Landau Problem in H^2 & Analytic Continuation

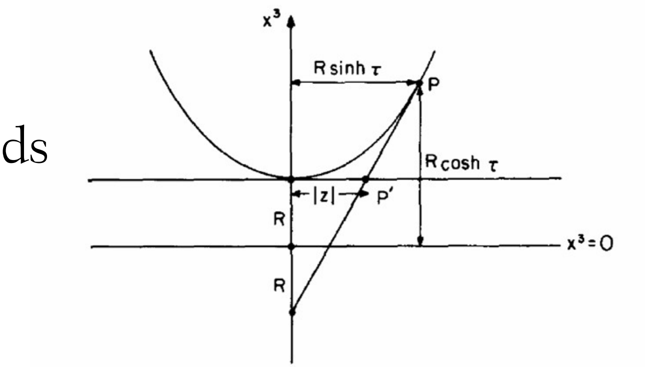
- Consider again charged scalar field in H^2 within a constant magnetic field [Comtet ’86; Dunne ‘91]

$$F = B\,\omega_{\mathbb{H}^2} = g\,\frac{d\tau_1\wedge d\tau_2}{\tau_2^2} \qquad \omega_{\mathbb{H}^2} = R^2\,\frac{d\tau_1\wedge d\tau_2}{\tau_2^2}$$

- The magnetic charge is no longer quantized and the Hamiltonian reads

$$H_{\mathbb{H}^2} = 2B\left(-\left(1-\frac{|z|^2}{2g}\right)^2\partial\bar{\partial}-\frac{1}{2}\left(1-\frac{|z|^2}{2g}\right)(z\partial-\bar{z}\bar{\partial})+\frac{1}{4}|z|^2\right)$$

in stereographic coordinates $z=\sqrt{2g}\,e^{i\theta}\tanh\left(\rho/2\right).$



- The relevant symmetry group is in this case SU(1,1), with generators [Dunne ‘91]

$$K_+ = -\frac{1}{\sqrt{2g}}z^2\partial+\sqrt{2g}\bar{\partial}-\sqrt{\frac{g}{2}}z \quad K_- = -\sqrt{2g}\partial+\frac{1}{\sqrt{2g}}\bar{z}^2\bar{\partial}-\sqrt{\frac{g}{2}}\bar{z} \quad K_0 = z\partial-\bar{z}\bar{\partial}+g$$

which verify $[K_+,K_-]=-2K_0\,,\quad [K_0,K_\pm]=\pm K_\pm$, such that $H_{\mathbb{H}^2} = B\,\left(C_2+g^2\right)/g$

Landau Problem in H^2 & Analytic Continuation

- The spectrum in this case is more complicated [Comtet, Houston ‘84; Comtet ’86]

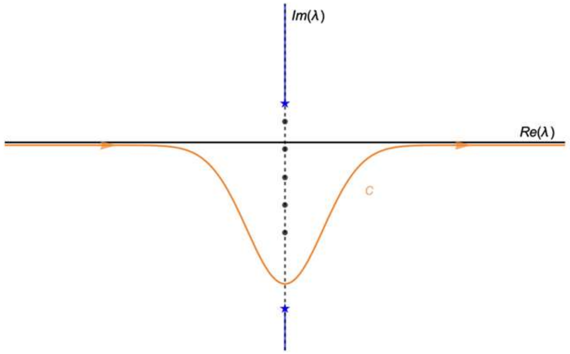
$$E_n = 2B\left(n+\frac{1}{2}-\frac{n(n+1)}{2g}\right) \qquad \text{with } \ 0 \leq n < g-\frac{1}{2}\,, \quad j \geq -n+g \qquad \leftsquigarrow \ \text{discrete (finite) states}$$

$$E_\lambda = \frac{B}{2g}\left(\frac{1}{4}+\lambda^2+g^2\right) \qquad \text{with } j=-\frac{1}{2}+i\lambda\,, \lambda \in \mathbb{R}_{\geq 0} \qquad \leftsquigarrow \ \text{principal (continuous) series}$$

- The relevant trace for the 1-loop determinant is [Comtet, Houston ‘84]

$$\mathrm{Tr}\,\left[e^{-\tau H_{\mathbb{H}^2}}\right] = \sum_{n=0}^{\lfloor g-\frac{1}{2}\rfloor} \rho_n\, e^{-\tau E_n} + \int_0^\infty d\lambda\, \rho_c(\lambda)\, e^{-\tau E_\lambda}$$

$$= -\frac{iV_{\mathbb{H}^2}}{(2\pi R)^2}\int_C d\lambda\,\lambda\left[\psi\left(\frac{1}{2}+i\lambda-g\right)+\psi\left(\frac{1}{2}+i\lambda+g\right)\right]\, e^{-\tau E_\lambda}$$



- Performing the analytic continuation $\tau_1 = it\,,\ g=-ie\,,\ H=-m^2$, we arrive at AdS$_2$ [Pioline, Troost ‘05]

$$\mathrm{Tr}\,\left[e^{-\tau H_{\mathrm{AdS}_2}}\right] = \int_0^\infty d\lambda\, \rho_E(\lambda)\, e^{-\tau E_\lambda} \qquad \text{with } \rho_E(\lambda) = \frac{V_{\mathrm{AdS}}}{2\pi R^2}\,\frac{\lambda\,\sinh(2\pi\lambda)}{\cosh(2\pi\lambda)+\cosh(2\pi e)}$$