

On Physical Discretization and Arithmetic Geometry

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Based On

hep-th/xxxx.xxxxx

Working Paper @ jjheckman.com/research

Disclaimers: Speculative Math and Physics

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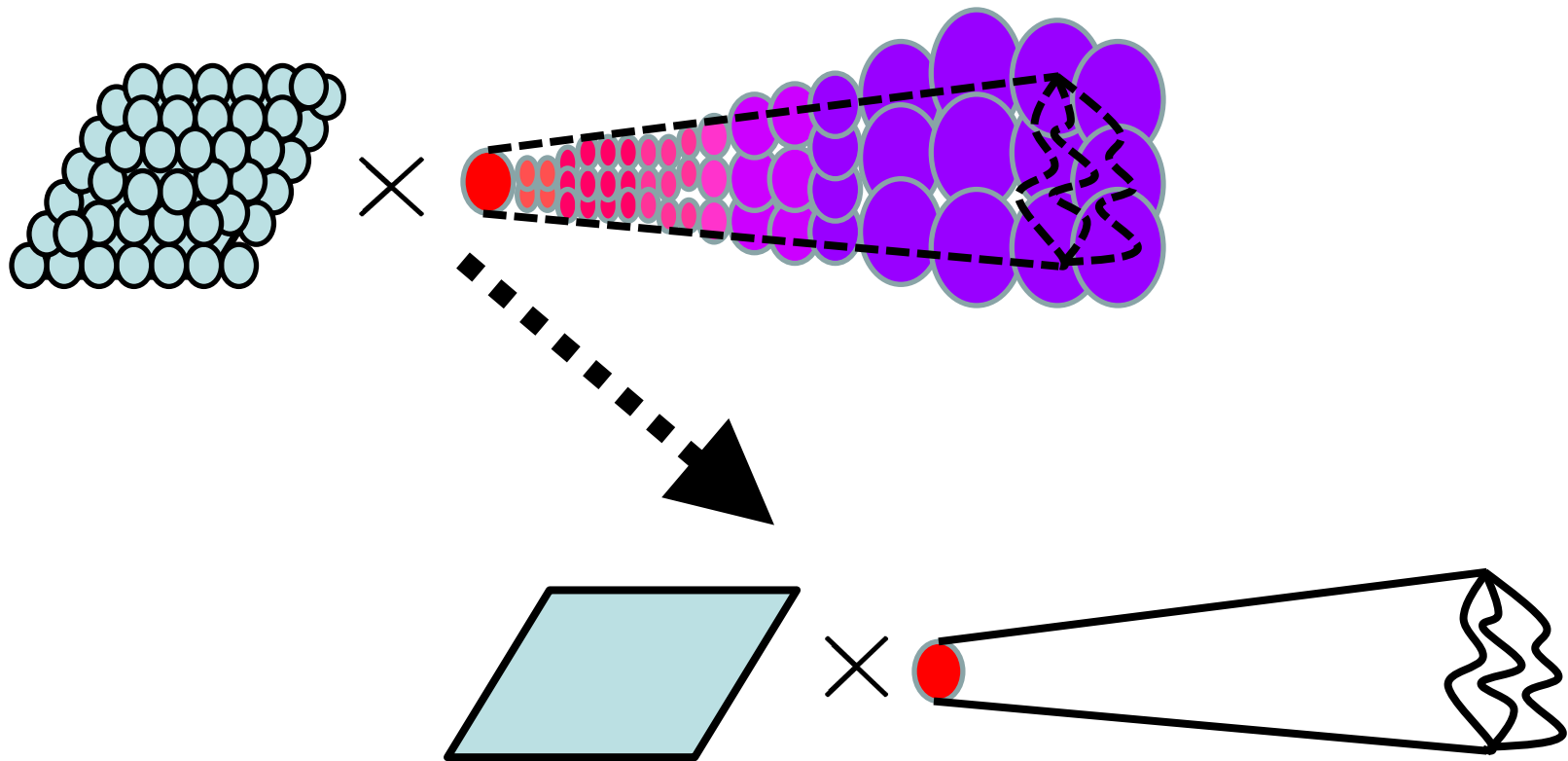
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Underlying Question

How to Define QFT?

How to Define Strings?

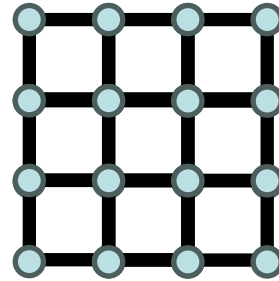


Motivation 1:

Discretizing physics is quite common:

Example: Lattice QFT

Discretize space / spacetime



Worries: Lorentz Breaking,
Chiral Degrees of Freedom,
What About Gravity,...

Motivation 2:

Flux vacua have interesting arithmetic!

Example:
$$W_{\text{GVW}} = \int_{\text{CY}_3} G \wedge \Omega$$

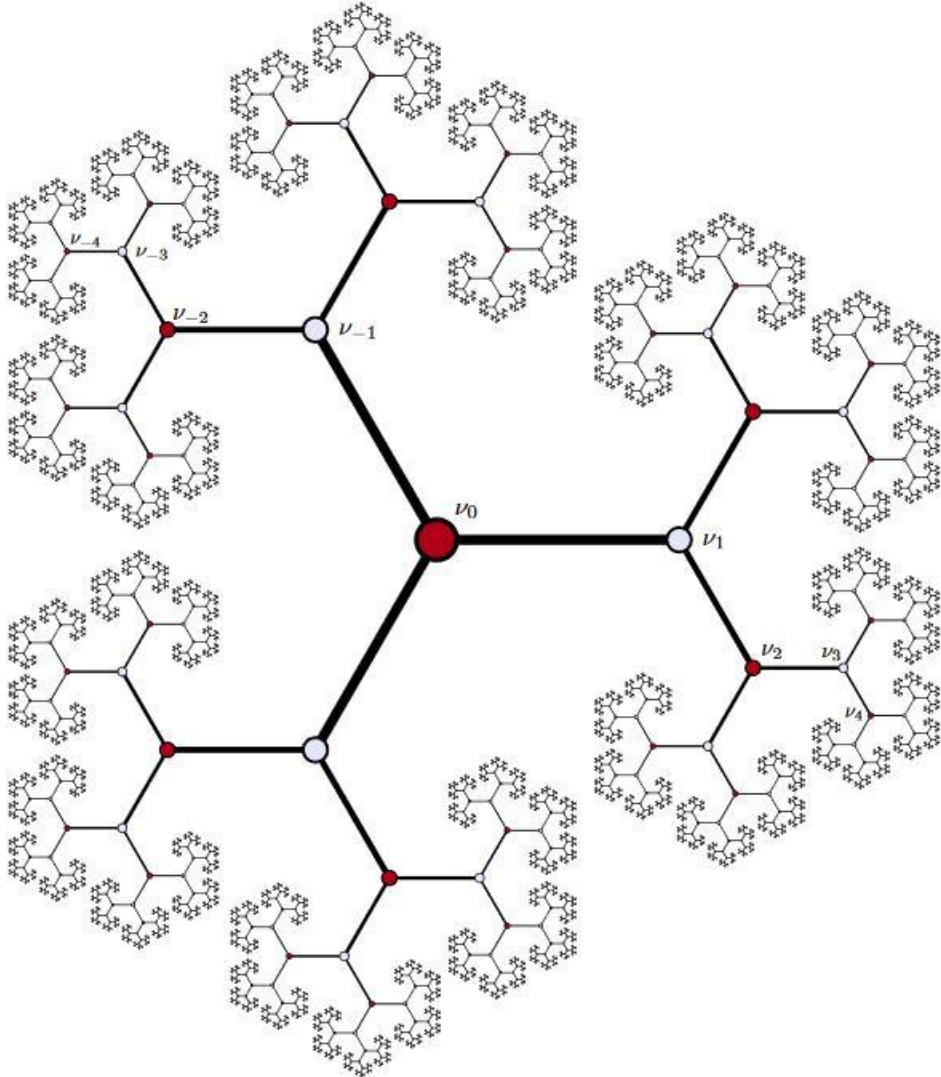
Stabilized CY cplx moduli *not* “random $\in \mathbb{C}$ ”

Far more rigid / arithmetic $\mathbb{Q}_p, \mathbb{C}_p, \dots$

Moore '98, Kanno Watari '17, Kachru Nally Wang '20,
Candelas de la Ossa Kuusela McGovern '23,...

Motivation 3:

p-adic AdS/CFT



Motivation 4:

\hbar limits of $\int [\mathcal{D}\phi] \exp(iS[\phi]/\hbar)$:

$\hbar \rightarrow 0$: Classical

$\hbar \rightarrow 1$: Quantum

$\hbar \rightarrow \infty$: ???

Example: BF theory with \mathbb{Z}_K fields:

$$S_{BF} = \frac{2\pi}{K} \int B \cup \delta A$$

Main Question / Idea

$$\hbar = \frac{N}{2\pi} \text{ with } N \in \mathbb{Z}_{>0}$$

Main Question / Idea

$$\hbar = \frac{N}{2\pi} \text{ with } N \in \mathbb{Z}_{>0}$$

¿QFT / Strings in this Regime?

$$iii\hbar = \frac{N}{2\pi} ???$$

$\hbar \rightarrow 0$: Classical Regime

$\hbar \rightarrow 1$: Quantum Regime

$\hbar \rightarrow \infty$: Arithmetic Regime

Punchline # 1

Common Theme in Strings and Geometry:

“Algebraic Geometry / \mathbb{C} ” (and sometimes \mathbb{R})

$\hbar = N/2\pi \rightarrow \infty$ Arithmetic Regime

“Algebraic Geometry / $\overline{\mathbb{F}}_p/\mathbb{Q}_p/\mathbb{C}_p, \dots$ ”

Punchline # 2

Lattice: Truncated $\dim \mathcal{H} < \infty$

Index Theorems?

$\hbar = N/2\pi \rightarrow \infty$ Arithmetic Regime

Spacetime truncated but \mathcal{H} NOT finite

Punchline # 3

p -adic strings revisited

“CFT” Correlators? $\exists T_{\mu\nu}^{\text{worldsheet}}$?

$\hbar = N/2\pi \rightarrow \infty$ Arithmetic Regime

Analytification: $\mathbb{Q}_p \rightarrow \text{Berkovich}$

Outline

- $\hbar = \frac{N}{2\pi} = \frac{p}{2\pi}$ & QFT in Char p
- $\hbar = \frac{N}{2\pi} = \frac{p^a}{2\pi}$ & QFT in $\mathbb{Q}_p/\mathbb{C}_p, \dots$
- p -adic Strings & Berkovich Strings
- Conclusions

QFT in Char p

QFT in Char p (Motivating Example)

QFT in Char p

(Motivating Example)

(Take $p \neq 2$ for simplicity: “2 is the oddest prime”)

1D Scalar

Consider Scalar Field Theory $\phi : \mathbb{R} \rightarrow \mathbb{R}$

$$S[\phi] = \int dt \frac{1}{2} (\partial_t \phi)^2 - V(\phi)$$

$$\mathcal{Z} = \int [\mathcal{D}\phi] \exp(iS[\phi]/\hbar)$$

$$\langle \phi(t_1) \dots \phi(t_m) \rangle = \frac{1}{\mathcal{Z}} \int [\mathcal{D}\phi] \exp(iS[\phi]/\hbar) \phi(t_1) \dots \phi(t_m)$$

Discretize

Discretize Source and Target $\phi : \mathbb{Z} \rightarrow \mathbb{Z}$

$$S[\phi] = \sum_t \frac{1}{2} (\phi_{t+1} - \phi_t)^2 - V(\phi_t)$$

$$\mathcal{Z} = \sum_{\phi: \mathbb{Z} \rightarrow \mathbb{Z}} \exp(iS[\phi]/\hbar)$$

$$\langle \phi(t_1) \dots \phi(t_m) \rangle = \frac{1}{\mathcal{Z}} \sum_{\phi: \mathbb{Z} \rightarrow \mathbb{Z}} \exp(iS[\phi]/\hbar) \phi(t_1) \dots \phi(t_m)$$

Set $\hbar = N/2\pi$

Discretize Source and Target $\phi : \mathbb{Z} \rightarrow \mathbb{Z}$

$$S[\phi] = \sum_t \frac{1}{2} (\phi_{t+1} - \phi_t)^2 - V(\phi_t)$$

$$\mathcal{Z} = \sum_{\phi: \mathbb{Z} \rightarrow \mathbb{Z}} \exp(2\pi i S[\phi]/N)$$

$$\langle \phi(t_1) \dots \phi(t_m) \rangle = \frac{1}{\mathcal{Z}} \sum_{\phi: \mathbb{Z} \rightarrow \mathbb{Z}} \exp(2\pi i S[\phi]/N) \phi(t_1) \dots \phi(t_m)$$

Coarse Truncation (Lattice)

(assume 2 invertible, i.e., it is in $(\mathbb{Z}/N\mathbb{Z})^\times$)

Mod N Source and Target $\phi : \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{Z}/N\mathbb{Z}$

$$S[\phi] = \sum_t \frac{1}{2} (\phi_{t+1} - \phi_t)^2 - V(\phi_t)$$

Operators: $e^{2\pi i \phi(t)/N}$

$$\langle \mathcal{O}(t_1) \dots \mathcal{O}(t_n) \rangle = \frac{1}{Z} \sum_{\phi: \mathbb{Z}_N \rightarrow \mathbb{Z}_N} e^{2\pi i S[\phi]/N} \mathcal{O}(t_1) \dots \mathcal{O}(t_n)$$

Less Coarse ($N = p$ prime)

Mod p poly: $\phi(x) = \sum_n \phi_n x^n$ with $\phi_n \in \mathbb{Z}/p\mathbb{Z}$

$$L[\phi] = \frac{1}{2}(\partial_x \phi(x))^2 - V(\phi(x))$$

Mod p Evaluation: $S[\phi] = \sum_t L[\phi]_{x=t \in \mathbb{Z}/p\mathbb{Z}}$

$$\langle \mathcal{O}(t_1) \dots \mathcal{O}(t_n) \rangle = \frac{1}{Z} \sum_{\phi \in \mathbb{F}_p[x]} e^{2\pi i S[\phi]/p} \mathcal{O}(t_1) \dots \mathcal{O}(t_n)$$

$$\phi(x) = \sum_n \phi_n x^n \text{ with } \phi_n \in \mathbb{Z}/p\mathbb{Z}$$

$\phi(x) \in \mathbb{F}_p[x]$ (\mathbb{F}_p (math) field with p elements)

Algebraic Geometry: Affine Line $\mathbb{A}_{\mathbb{F}_p}^1 = \text{Spec } \mathbb{F}_p[x]$

Morphism $\phi : \mathbb{A}_{\mathbb{F}_p}^1 \dashrightarrow \mathbb{A}_{\mathbb{F}_p}^1$

Reminder: $\mathbb{A}_{\mathbb{C}}^1 = \text{Spec } \mathbb{C}[x]$

Point \leftrightarrow Ideal $\langle x - c \rangle$

QFT in Char p

Physical Fields: $\phi : X \dashrightarrow Y$

polynomials / power series in local coordinates

$\#$ Points $< \infty$ BUT $\#$ maps $= \infty$

\Rightarrow More data than lattice truncation

Characteristic p Geometry

Instead of Geometry over \mathbb{C} ,

“Characteristic p ” i.e., “ $p = 0$ ”

Geometry over $\mathbb{F}_p, \mathbb{F}_{q=p^n}, \overline{\mathbb{F}}_p, \dots$

Characteristic p Geometry

Instead of Geometry over \mathbb{C} ,

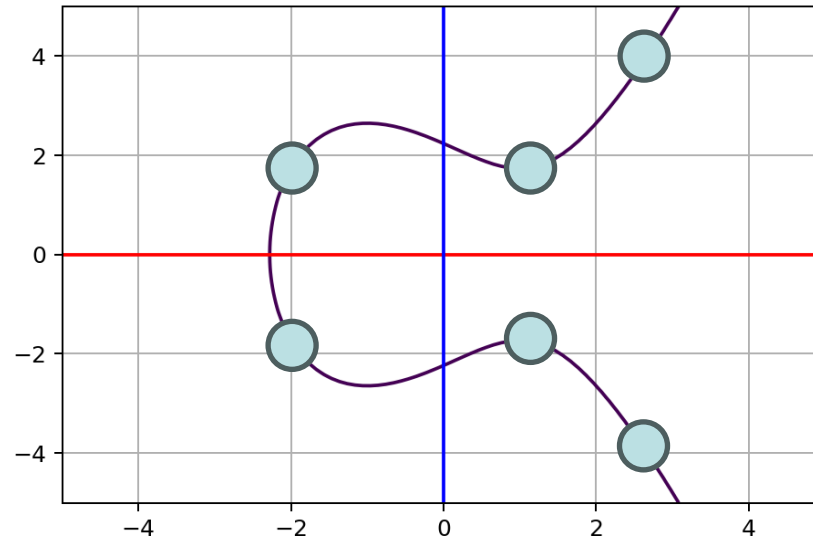
“Characteristic p ” i.e., “ $p = 0$ ”

Geometry over $\mathbb{F}_p, \mathbb{F}_{q=p^k}, \overline{\mathbb{F}}_p, \dots$

Look for zero sets: $\phi(x, y) = \sum_{m,n} \phi_{m,n} x^m y^n = 0$

Discretization: $\#$ Points $< \infty$ for \mathbb{F}_p and \mathbb{F}_q

Characteristic p Geometry



The machinery of algebraic geometry carries over

This still defines a genus 1 curve: $y^2 = x^3 + fx + g$

More Geometrically...

Fix X_{source} and Y_{target} varieties / \mathbb{F}_p

Consider $\phi : X \dashrightarrow Y$ rational (allow poles)

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle \equiv \frac{1}{\mathcal{N}} \sum_{\phi: X \dashrightarrow Y} \exp\left(\frac{2\pi i}{p} S[\phi]\right) \mathcal{O}_1 \dots \mathcal{O}_n$$

What About $/ \mathbb{F}_q$?

Clear interpretation for $Y_{\text{target}}/\mathbb{F}_q$: add more fields
($\text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ a “global symmetry”)

Example: $\mathbb{F}_q = \mathbb{F}_p(\omega)$ with $q = p^2$.

Locally, $\phi = \phi_1\omega + \phi_2\omega^p \in \mathbb{F}_q[x]$

Note: X_{source} can often also be viewed as a target space

Path Integrals / \mathbb{F}_q

Fix X_{source} and Y_{target} varieties / \mathbb{F}_q

Consider $\phi : X \dashrightarrow Y$ rational (allow poles)

Still demand $S[\phi] \in \mathbb{F}_p$ (Unitarity)

(Can enforce by adding Frobenius conjugates)

$$x \mapsto x^p$$

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle \equiv \frac{1}{\mathcal{Z}} \sum_{\phi: X \dashrightarrow Y} \exp\left(\frac{2\pi i}{p} S[\phi]\right) \mathcal{O}_1 \dots \mathcal{O}_n$$

Formal limits available $\mathbb{F}_p \subset \mathbb{F}_{p^2} \subset \dots \subset \overline{\mathbb{F}}_p$

Hilbert Space

Subtleties with States

Consider $\phi : \mathbb{A}_{\text{time}}^1 \rightarrow \mathbb{A}_{\text{target}}^1$ (over \mathbb{F}_q)

We can define $|\phi : \mathbb{A}^1 \rightarrow \mathbb{A}^1\rangle \in \mathcal{H}_{\text{big}}$ a Hilbert space / \mathbb{C}
“Explicit Time Dependence”

We can also define $|\phi\rangle \in \mathcal{H}_{\text{small}}$ a Hilbert space / \mathbb{C}
“Lattice Approximation” projects $\mathbb{A}^1 \rightarrow \mathbb{F}_q$

Path integral still computes expectation values

Supersymmetry & Cohomology

A Supersymmetric Theory

Introduce $W(\phi)$ a “superpotential” (poly in ϕ)

$$L = \frac{1}{2}(\partial_t\phi)^2 + \hat{i}\chi\partial_t\psi - \frac{\hat{i}^2}{2}f^2 + W'f + \hat{i}W''\chi\psi$$

$\delta_{\text{SUSY}}L = \text{exact differential}$

$$Q_+ = \hat{i}\psi \left(\frac{\partial}{\partial\phi} + \hat{i}^{-1} \frac{\partial W}{\partial\phi} \right) = - \left(\partial_t\phi + \hat{i}^{-1} \frac{\partial W}{\partial\phi} \right) \frac{\partial}{\partial\chi}$$

$$Q_- = \hat{i}\chi \left(\frac{\partial}{\partial\phi} - \hat{i}^{-1} \frac{\partial W}{\partial\phi} \right) = - \left(\partial_t\phi - \hat{i}^{-1} \frac{\partial W}{\partial\phi} \right) \frac{\partial}{\partial\psi}$$

A Cohomology Theory (I)

$Q^2 = 0 \Rightarrow$ cohomology theory

Question: Is H_Q^\bullet related to known H^\bullet 's?

Answer (1): In char 0, we would say de Rham cohomology

Question: What about in char p ?

A Cohomology Theory (II)

Note: Our construction also makes sense over $\mathbb{Z}/p^n\mathbb{Z}$

$\Rightarrow H_Q^\bullet$ computed via inverse limit

Proposal: $H_Q^\bullet \simeq H_{\text{cris}}^\bullet$ (in smooth case)
(Grothendieck '66; Berthelot '74;...)

Proposal: $H_Q^\bullet \simeq H_{\text{rig}}^\bullet$ (more generally)
(Berthelot '86; Kedlaya '06;...)

An Index

Witten Index $\mathrm{Tr}(-1)^{\mathbf{F}} = \mathrm{Tr}_{\mathbf{F}\mathbf{r}}(\dim\ker Q - \dim\mathrm{coker} Q)$
(Witten '82)

Same setup as Hasse-Weil Zeta Function: (Weil '49)

$$\log Z_{V, \mathbb{F}_q}(z) = \sum_n \#V(\mathbb{F}_{q^n}) \frac{z^n}{n} = \sum_n \mathrm{Tr}_n(-1)^{\mathbf{F}} \frac{z^n}{n}$$

(can use étale ℓ -adic or rigid cohomology)

(see Kedlaya '06)

Counting Points

(Le Stum Review)

$$Z_{\mathbb{A}^n, q}(z) = \frac{1}{1 - q^n z}$$

$$Z_{\mathbb{P}^n, q}(z) = \frac{1}{(1-z)(1-qz)\dots(1-q^n z)}$$

$$Z_{\mathbb{E}, q}(z) = \frac{1 - az + qz^2}{(1-z)(1-qz)} = (1 - az + qz^2)Z_{\mathbb{P}^1, q}(z)$$

$$\#\mathbb{E}(\mathbb{F}_q) = -a + 1 + q$$

$$\mathbb{E} : \{Y^2 Z = X^3 + XZ^2\} \subset \mathbb{P}^3$$

Limits: $\hbar = \frac{p^a}{2\pi}$ and $a \rightarrow \infty$

¿Make Sense Of?

Path Integral: $\int [\mathcal{D}\phi] \exp(2\pi i S[\phi]/p^a)$:

Expand $u \in \mathbb{Z}$ as: $u = \sum_m u_m p^m$ “ p -adic expansion”

Approx 1: Reduce mod p

Approx 2: Reduce mod p^2

-
-
-

Approx a : Reduce mod p^a

Building $\mathbb{Z}/p^a\mathbb{Z}$ Actions

Introduce $\phi(x) = \sum_n \phi_n x^n = \sum_{n,m} \phi_{n,m} x^n p^m$

Build (truncated) actions as before:

$$S[\phi] = \sum_{\text{pts}} \frac{1}{2} (\partial_x \phi)^2 - V(\phi)|_{x=\text{pt}}$$

Limits

$$\mathbb{Q}_p \text{ Limit: } \int [\mathcal{D}\phi] \exp\left(\lim_{a \rightarrow \infty} 2\pi i S[\phi]/p^a\right)$$
$$\phi : X(\mathbb{Q}_p) \dashrightarrow Y(\mathbb{Q}_p)$$

$$\mathbb{R} \text{ Limit: } \lim_{a \rightarrow \infty} \int [\mathcal{D}\phi] \exp(2\pi i S[\phi]/p^a)$$
$$\phi : X(\mathbb{Q}_p) \dashrightarrow Y(\mathbb{R})$$

p -adic & Berkovich String

Worldsheet \rightarrow Target

Open p -adic string at tree level (80's version)
(Freund Olson '87, Freund Witten '87)

$$\phi^A : \mathbb{Q}_p \rightarrow \mathbb{R}^{9,1}$$

$$S[\phi] = \int_{\mathbb{Q}_p} dx G_{AB} \phi^A \mathcal{D}_{\text{Vld}} \phi^B$$

$$\sim \int_{\mathbb{Q}_p} dx G_{AB} \mathcal{D}_{\text{Vld}} \phi^A \mathcal{D}_{\text{Vld}} \phi^B$$

Questions / Issues

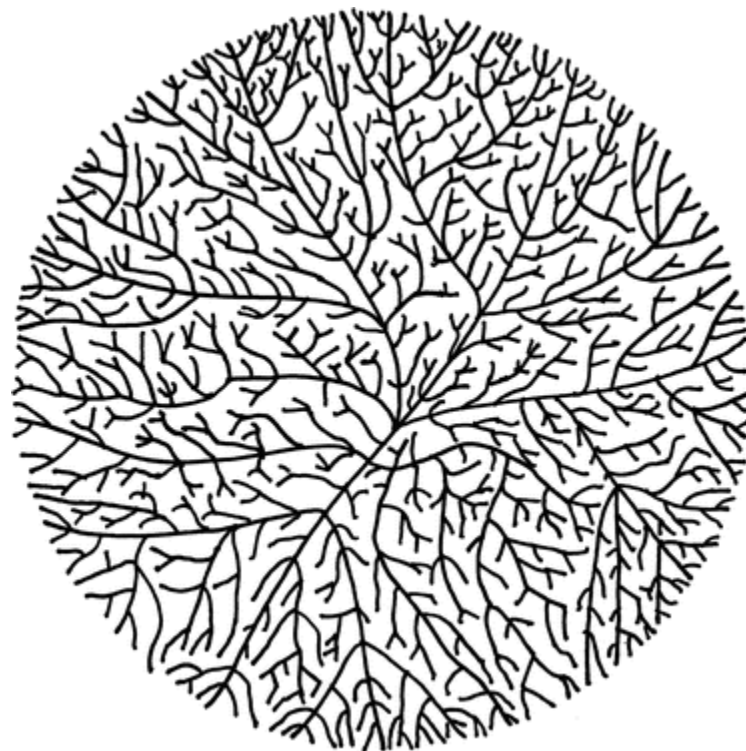
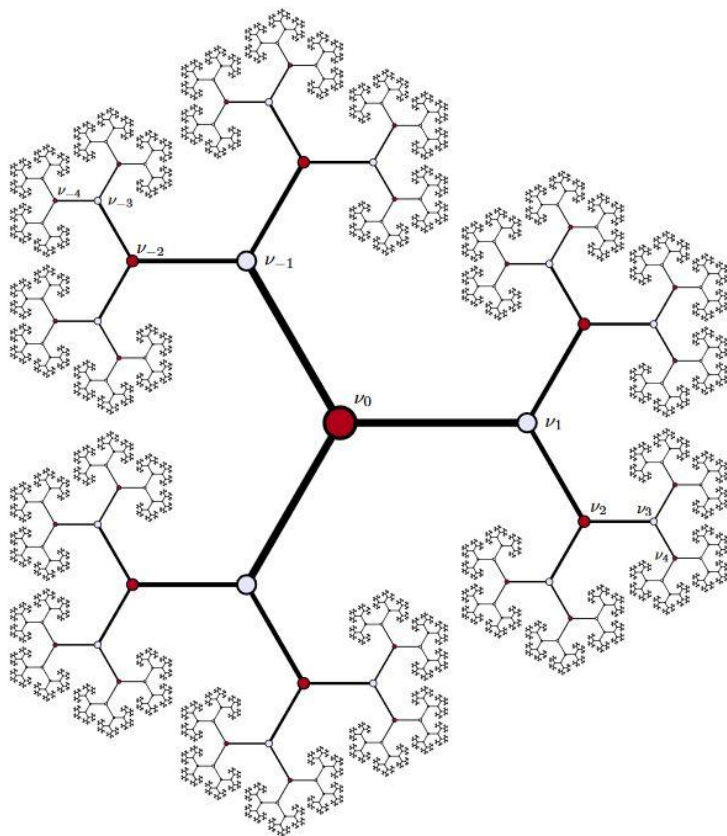
Open p -adic string at tree level (80's version)
(Freund Olson '87, Freund Witten '87)

Questions:

- Beyond Tree Level?
- Closed Strings?
- Worldsheet Stress Tensor $T_{\mu\nu}$?
- \mathbb{Q}_p Very Coarse...

$\mathbb{Q}_p \rightarrow$ Berkovich Space

Berkovich '90



Berkovich Strings

Berkovich space smoother

$\exists d', d''$ operators and “ $\partial\bar{\partial}$ Lemma”:

Chambert-Loir Ducros '12

Green's Function: $d' d'' G(x, y) = \delta_{\text{pt}}(x, y)$

Example Action: $\int_{\mathfrak{B}} d\text{Vol}_{\mathfrak{B}} d' \phi d'' \phi$

Stress Tensor: $T_{\text{“hol”}} = d' \phi d' \phi$

JJH In Progress

Berkovich Tachyon Amplitude

$$\mathfrak{M}_{2 \rightarrow 2}^{(p)} = \prod_{x=s,t,u} \frac{1}{1-p^{\alpha(x)}} \quad \text{with } \alpha(x) \equiv 1 + \frac{1}{2}x$$

$$\mathfrak{M}_{\text{adelic}} = \prod_{p \text{ prime}} \mathfrak{M}_p = \prod_{x=s,t,u} \zeta(-\alpha(x))$$

$$\mathfrak{M}_{\text{phys}} = \frac{1}{\mathfrak{M}_{\text{adelic}}} \quad \text{Poles at } \zeta \text{ zeroes...}$$

Summary / Future

What Was This Talk About

- Arithmetic Regime: $\hbar = N/2\pi$
- Building QFT / $\mathbb{F}_p, \overline{\mathbb{F}}_p, \mathbb{Q}_p, \dots$
Beyond Lattice Truncations
- Analytification / Smoothing of p -adic string
- Many Topics Not Mentioned...

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