CLUSTER OF EXCELLENCE
QUANTUM UNIVERSE

CRC 1624

Algebro-Geometric Bootstrapping from OPE Decoupling

Craig Lawrie (DESY)

Based on: 2507.14532 w/ M.J. Kang and J. Song
2602.04946 w/ M.J. Kang and J. Song
260n.nnnnn w/ D. Annamalai, T. Hussel, M.J. Kang, and J. Song
260n.nnnnn w/ M.J. Kang and J. Song

Strings & Geometry 2026

19-05-2026

Motivation

4d $\mathcal{N}=2$ SCFTs, like all eight-supercharge theories, have much interesting physics

Motivation

4d $\mathcal{N} = 2$ SCFTs, like all eight-supercharge theories, have much interesting physics

(local) SCFT is
defined by CFT data

$$\{\Delta_i, \lambda_{ijk}\}$$

2-point
correlation functions

3-point
correlation functions

Motivation

4d $\mathcal{N} = 2$ SCFTs, like all eight-supercharge theories, have much interesting physics

(local) SCFT is
defined by CFT data

$$\{\Delta_i, \lambda_{ijk}\}$$

2-point
correlation functions

3-point
correlation functions

$\rightsquigarrow$

CFT data follows
from 1/4-BPS Schur
operator sector

$$\mathbb{V}$$

vertex operator algebra

lore

Motivation

4d $\mathcal{N} = 2$ SCFTs, like all eight-supercharge theories, have much interesting physics

(local) SCFT is defined by CFT data

$$\{\Delta_i, \lambda_{ijk}\}$$

2-point correlation functions

3-point correlation functions

$\rightsquigarrow$

CFT data follows from 1/4-BPS Schur operator sector

$$\mathbb{V}$$

vertex operator algebra

lore

Strings & Geometry

Motivation

4d $\mathcal{N} = 2$ SCFTs, like all eight-supercharge theories, have much interesting physics

(local) SCFT is defined by CFT data

$$\{\Delta_i, \lambda_{ijk}\}$$

2-point correlation functions

3-point correlation functions

$\rightsquigarrow$

CFT data follows from 1/4-BPS Schur operator sector

$$\mathbb{V}$$

vertex operator algebra

$\rightsquigarrow$

(all?) information of the VOA follows from ?

$$R$$

bifiltered ring "over" the Higgs branch

lore

today

Spoiler: The Macdonald Index from Geometry

[Kang, CL, Song]

Proposal [KLS]: For each 4d $\mathcal{N}=2$ SCFT, there exists a bifiltered ring R such that

$$I_{\mathrm{Mac}}(q,T) = \mathrm{HS}_{q,q,T}\left(\mathrm{gr}(J_\infty(R))\right)$$

and

$$\mathcal{H} = \mathrm{Spec}(R)_{\mathrm{red}}\,.$$

Spoiler: The Macdonald Index from Geometry

[Kang, CL, Song]

Proposal [KLS]: For each 4d $\mathcal{N} = 2$ SCFT, there exists a bifiltered ring R such that

$$I_{\text{Mac}}(q, T) = \text{HS}_{q,q,T}\left(\text{gr}(J_\infty(R))\right)$$

and

$$\mathcal{H} = \text{Spec}(R)_{\text{red}}\,.$$

Goal: explain what each of these symbols mean

give evidence that they hold for certain (Argyres--Douglas) SCFTs

Spoiler: The Macdonald Index from Geometry

[Kang, CL, Song]

Proposal [KLS]: For each 4d $\mathcal{N} = 2$ SCFT, there exists a bifiltered ring R such that

$$I_{\text{Mac}}(q, T) = \text{HS}_{q,q,T}\left(\text{gr}(J_\infty(R))\right)$$

and

$$\mathcal{H} = \text{Spec}(R)_{\text{red}}\,.$$

counting
protected,
1/4-BPS operators

$\text{Spec}(R)$ = an affine (bi-filtered) scheme which reduces to the Higgs branch

Explaining the Proposal

$$\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_\infty(R)))$$

$\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_\infty(R)))$

Bifiltered Rings: A Simple Construction

a simple way to construct an explicit bifiltered ring is

let $S = \mathbb{C}[x_1, \cdots, x_n]$ be the bigraded ring with bigrading $\deg(x_i) = (a_i, b_i)$ $\longrightarrow$ $S = \bigoplus_{p,q \in \mathbb{Z}} S_{p,q}$

$\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_\infty(R)))$

Bifiltered Rings: A Simple Construction

a simple way to construct an explicit bifiltered ring is

let $S = \mathbb{C}[x_1, \cdots, x_n]$ be the bigraded ring with bigrading $\deg(x_i) = (a_i, b_i)$ $\longrightarrow$ $S = \bigoplus_{p,q \in \mathbb{Z}} S_{p,q}$

let I be a not necessarily bihomogeneous ideal of S

$\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_\infty(R)))$

Bifiltered Rings: A Simple Construction

a simple way to construct an explicit bifiltered ring is

let $S = \mathbb{C}[x_1, \cdots, x_n]$ be the bigraded ring with bigrading $\deg(x_i) = (a_i, b_i)$ $\longrightarrow$ $S = \bigoplus_{p,q\in\mathbb{Z}} S_{p,q}$

let I be a not necessarily bihomogeneous ideal of S

then, S/I is *not* bigraded since I mixes degrees

Bifiltered Rings: A Simple Construction

a simple way to construct an explicit bifiltered ring is

let $S = \mathbb{C}[x_1, \cdots, x_n]$ be the bigraded ring with bigrading $\deg(x_i) = (a_i, b_i)$ ⟶ $S = \bigoplus_{p,q \in \mathbb{Z}} S_{p,q}$

let I be a not necessarily bihomogeneous ideal of S

then, S/I is *not* bigraded since I mixes degrees

example: $R = \mathbb{C}[x, y]/(x^2 + y^3)$ with $\deg(x) = (3,2)$, $\deg(y) = (2,1)$

bi-degree $(6,4)$ bi-degree $(6,3)$

Bifiltered Rings: A Simple Construction

a simple way to construct an explicit bifiltered ring is

let $S = \mathbb{C}[x_1, \cdots, x_n]$ be the bigraded ring with bigrading $\deg(x_i) = (a_i, b_i)$ $\longrightarrow$ $S = \bigoplus_{p,q \in \mathbb{Z}} S_{p,q}$

let I be a not necessarily bihomogeneous ideal of S

then, S/I is *not* bigraded since I mixes degrees

a bifiltration on S/I is $F_{p,q}R =$ image of $\left(\bigoplus_{p' \le p, q' \le q} S_{p',q'} \right)$ in R

example: $R = \mathbb{C}[x,y]/(x^2 + y^3)$ with $\deg(x) = (3,2)$, $\deg(y) = (2,1)$

bi-degree $(6,4)$ bi-degree $(6,3)$

we will use this construction ubiquitously!

note: the Hilbert series is defined for *graded* rings, not *filtered* rings

$\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_\infty(R)))$

What is an... Associated Graded Ring

a graded ring has a decomposition as $R = \bigoplus_{p\in\mathbb{Z}} R_p$

from a bifiltered ring, we can construct an associated bigraded ring, $\mathrm{gr}(R)$

$\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_\infty(R)))$

What is an... Associated Graded Ring

a graded ring has a decomposition as $R = \bigoplus_{p \in \mathbb{Z}} R_p$

from a bifiltered ring, we can construct an associated bigraded ring, $\mathrm{gr}(R)$

define $\mathrm{gr}_{p,q}(R) = \dfrac{F_{p,q}R}{F_{p-1,q}R \oplus F_{p,q-1}R}$ then $\mathrm{gr}(R) = \bigoplus_{p,q} \mathrm{gr}_{p,q}(R)$ is bigraded

example: $R = \mathbb{C}[x,y]$

$F_{p,q}R = \{$ polynomials $P(x,y)$ with $\deg_{x,y}(P) \leq (p,q)$ $\}$

$\mathrm{gr}_{p,q}(R) = \{$ polynomials $P(x,y)$ with $\deg_{x,y}(P) = (p,q)$ $\}$

$\Rightarrow$ $\mathrm{gr}(R) = \mathbb{C}[x,y]$ with $\deg(x) = (1,0)$, $\deg(y) = (0,1)$

$\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_\infty(R)))$

What is a... Jet Scheme

consider a polynomial ring:

arbitrary polynomials (homogeneous or inhomogeneous) in $x_1, \cdots, x_m$

$$R = \mathbb{C}[x_1, x_2, \cdots, x_m] / (f_1, f_2, \cdots, f_r)$$

no assumption of filtration or grading

$\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_\infty(R)))$

What is a... Jet Scheme

consider a polynomial ring:

arbitrary polynomials (homogeneous or inhomogeneous) in $x_1, \cdots, x_m$

$$R = \mathbb{C}[x_1, x_2, \cdots, x_m] / (f_1, f_2, \cdots, f_r)$$

no assumption of filtration or grading

formally expand as a power series: $x_j \to x_j(t) = \sum_{\alpha=0}^{n} x_j^{(\alpha)} t^\alpha$

$$f_i \mod t^{n+1} = \sum_{\alpha=0}^{n} f_i^{(\alpha)} t^\alpha$$

$\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_\infty(R)))$

What is a... Jet Scheme

consider a polynomial ring:

arbitrary polynomials (homogeneous or inhomogeneous) in $x_1, \cdots, x_m$

$$R = \mathbb{C}[x_1, x_2, \cdots, x_m] / (f_1, f_2, \cdots, f_r)$$

no assumption of filtration or grading

formally expand as a power series: $x_j \to x_j(t) = \sum_{\alpha=0}^{n} x_j^{(\alpha)} t^\alpha$

$$f_i \mod t^{n+1} = \sum_{\alpha=0}^{n} f_i^{(\alpha)} t^\alpha$$

then, we can define a new ring

$$J_n(R) = \mathbb{C}[x_1^{(0)}, \cdots, x_1^{(n)}, \cdots, x_m^{(0)}, \cdots, x_m^{(n)}] / (f_1^{(0)}, \cdots, f_1^{(n)}, \cdots, f_r^{(0)}, \cdots, f_r^{(n)})$$

if $X = \mathrm{Spec}(R)$ then $J_n(R)$ is the coordinate ring of the nth jet scheme of X: $J_n(X)$

$\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_\infty(R)))$

What is a... Jet Scheme

then, we can define a new ring

$$J_n(R) = \mathbb{C}[x_1^{(0)}, \cdots, x_1^{(n)}, \cdots, x_m^{(0)}, \cdots, x_m^{(n)}] / (f_1^{(0)}, \cdots, f_1^{(n)}, \cdots, f_r^{(0)}, \cdots, f_r^{(n)})$$

if $X = \mathrm{Spec}(R)$ then $J_n(R)$ is the coordinate ring of the nth jet scheme of X: $J_n(X)$

$J_n(R)$ is naturally a graded ring with "jet" grading $\deg(x_j^{(\alpha)}) = \alpha$

further: if R is bifiltered, $J_n(R)$ inherits the bifiltration

inherited bifiltration + jet grading defines a trifiltration on $J_n(R)$

$\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_\infty(R)))$

What is a... Jet Scheme

then, we can define a new ring

$$J_n(R) = \mathbb{C}[x_1^{(0)}, \cdots, x_1^{(n)}, \cdots, x_m^{(0)}, \cdots, x_m^{(n)}] \,/\, (f_1^{(0)}, \cdots, f_1^{(n)}, \cdots, f_r^{(0)}, \cdots, f_r^{(n)})$$

if $X = \mathrm{Spec}(R)$ then $J_n(R)$ is the coordinate ring of the nth jet scheme of X: $J_n(X)$

$J_n(R)$ is naturally a graded ring with "jet" grading $\deg(x_j^{(\alpha)}) = \alpha$

further: if R is bifiltered, $J_n(R)$ inherits the bifiltration

inherited bifiltration + jet grading defines a trifiltration on $J_n(R)$

the coordinate ring of the arc space occurs when we take the limit

$$J_\infty(R) = \lim_{\substack{\rightarrow \\ n}} J_n(R)$$

Algebro-Geometric Bootstrapping

[Kang, CL, Song]

$$\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_\infty(R)))$$

Algebro-Geometric Bootstrapping

[Kang, CL, Song]

$$\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_\infty(R)))$$

bi-filtered

Algebro-Geometric Bootstrapping

[Kang, CL, Song]

$$\mathrm{HS}_{q,q,T}(\mathrm{gr}(\underbrace{J_\infty(\underset{\text{bi-filtered}}{R})}_{\text{tri-filtered}}))$$

Algebro-Geometric Bootstrapping

[Kang, CL, Song]

$$\mathrm{HS}_{q,q,T}\Big(\underbrace{\mathrm{gr}\big(\underbrace{J_\infty(\underbrace{R}_{\texttt{bi-filtered}})}_{\texttt{tri-filtered}}\big)}_{\texttt{tri-graded}}\Big)$$

Algebro-Geometric Bootstrapping

[Kang, CL, Song]

$$\mathrm{HS}_{\underbrace{q}_{\text{jet-grading}},q,T}\Big(\underbrace{\mathrm{gr}\big(\underbrace{J_\infty(\underbrace{R}_{\text{bi-filtered}})}_{\text{tri-filtered}}\big)}_{\text{tri-graded}}\Big)$$

Algebro-Geometric Bootstrapping

[Kang, CL, Song]

$$\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_\infty(R)))$$

jet-grading

bi-filtered

tri-filtered

tri-graded

challenges: (technical) $J_\infty(R)$ has infinity-many generators and relations

(technical) $J_\infty(R)$ has non-trivial syzygies → must determine Groebner basis

(conceptual) how do we determine R for a given SQFT?

How to Find the Bifiltered Rings

we propose: "decoupling" in the space of CFT data

⇒ captures foundational information about the CFT

OPE of local operators $$O \times O' = \sum_{O''} \lambda_{OO'O''} O''$$

How to Find the Bifiltered Rings

we propose: "decoupling" in the space of CFT data

⇒ captures foundational information about the CFT

OPE of local operators

$$O \times O' = \sum_{O''} \lambda_{OO'O''} O''$$

generically, all O'' compatible with symmetries

sometimes, we discover that an operator O'' is absent: "decoupling"

How to Find the Bifiltered Rings

we propose: "decoupling" in the space of CFT data

⇒ captures foundational information about the CFT

OPE of local operators $$O \times O' = \sum_{O''} \lambda_{OO'O''} O''$$

generically, all O'' compatible with symmetries

sometimes, we discover that an operator O'' is absent: "decoupling"

Question: how much information of the theory can be recovered from OPE decoupling?

How to Find the Bifiltered Rings

we propose: "decoupling" in the space of CFT data

$\Rightarrow$ captures foundational information about the CFT

often, short multiplets that would generically appear in OPEs of universal operators are, in fact, absent [Agarwal, Lee, Song] [Liendo, Ramirez, Seo]

How to Find the Bifiltered Rings

we propose: "decoupling" in the space of CFT data

$\Rightarrow$ captures foundational information about the CFT

often, short multiplets that would generically appear in OPEs of universal operators are, in fact, absent [Agarwal, Lee, Song] [Liendo, Ramirez, Seo]

example: OPE of stress-tensor $T^{n+1} = (\widehat{C}_{0(0,0)})^{n+1}$ normally contains $\widehat{C}_{n(n/2,n/2)}$

How to Find the Bifiltered Rings

we propose: "decoupling" in the space of CFT data

$\Rightarrow$ captures foundational information about the CFT

often, short multiplets that would generically appear in OPEs of universal operators are, in fact, absent [Agarwal, Lee, Song] [Liendo, Ramirez, Seo]

example: OPE of stress-tensor $T^{n+1} = (\widehat{C}_{0(0,0)})^{n+1}$ normally contains $\widehat{C}_{n(n/2,n/2)}$

for the (A_1, A_{2n}) Argyres--Douglas theories, this multiplet is absent

How to Find the Bifiltered Rings

we propose: "decoupling" in the space of CFT data

⇒ captures foundational information about the CFT

often, short multiplets that would generically appear in OPEs of universal operators are, in fact, absent [Agarwal, Lee, Song] [Liendo, Ramirez, Seo]

example: OPE of stress-tensor $T^{n+1} = (\widehat{C}_{0(0,0)})^{n+1}$ normally contains $\widehat{C}_{n(n/2,n/2)}$

for the (A_1, A_{2n}) Argyres--Douglas theories, this multiplet is absent

this can be translated into the algebraic condition $x^{n+1} = 0$

How to Find the Bifiltered Rings

decoupling in OPEs of 4d Schur operators can be found by
studying null relations in the vertex operator algebra
under the SCFT $\leftrightarrow$ VOA correspondence [Beem, Lemos, Liendo, Peelaers, Rastelli, van Rees]

How to Find the Bifiltered Rings

decoupling in OPEs of 4d Schur operators can be found by studying null relations in the vertex operator algebra under the SCFT ↔ VOA correspondence [Beem, Lemos, Liendo, Peelaers, Rastelli, van Rees]

for (A_1, A_{2n}) the associated VOA is $M(2,2n+3)$ which contains a null state at level $2n+2$ of the schematic form

$$((L_{-2})^{n+1} + \cdots)\,\Omega$$

strong generator

vacuum state

How to Find the Bifiltered Rings

decoupling in OPEs of 4d Schur operators can be found by studying null relations in the vertex operator algebra under the SCFT ↔ VOA correspondence [Beem, Lemos, Liendo, Peelaers, Rastelli, van Rees]

for (A_1, A_{2n}) the associated VOA is $M(2,2n+3)$ which contains a null state at level $2n+2$ of the schematic form

$$((L_{-2})^{n+1} + \cdots)\,\Omega$$

strong generator

vacuum state

⇒ studying null relations in the VOA → polynomial relations in the strong generators → a candidate ring R

Examples: Higgs branch = point

Example: The (A_2, A_3) Argyres--Douglas SCFT

[Kang, CL, Song]

the VOA associated to (A_{k-1}, A_{n-1}) with $\gcd(k,n) = 1$ is a W-algebra minimal model [Buican, Nishinaka], [Cordova, Shao]

Example: The (A_2, A_3) Argyres--Douglas SCFT

[Kang, CL, Song]

the VOA associated to (A_{k-1}, A_{n-1}) with $\gcd(k,n)=1$ is a W-algebra minimal model [Buican, Nishinaka], [Cordova, Shao]

we are interested in the W_3-algebra, which has been studied in [Agarwal, Lee, Song]

Example: The (A_2, A_3) Argyres--Douglas SCFT

[Kang, CL, Song]

the VOA associated to (A_{k-1}, A_{n-1}) with $\gcd(k,n)=1$ is a W-algebra minimal model [Buican, Nishinaka], [Cordova, Shao]

we are interested in the W_3-algebra, which has been studied in [Agarwal, Lee, Song]

Level 5

$$\left(W_{-3}L_{-2} - \frac{10}{7}W_{-5}\right)\left|\Omega, c = -\frac{114}{7}\right\rangle \tag{B.3}$$

Level 6

$$\left(\frac{102}{49}L_{-6} - \frac{24}{7}L_{-4}L_{-2} - \frac{6}{7}(L_{-3})^2 + (L_{-2})^3 - \frac{39}{7}(W_{-3})^2\right)\left|\Omega, c = -\frac{114}{7}\right\rangle \tag{B.4}$$

Example: The (A_2, A_3) Argyres--Douglas SCFT

[Kang, CL, Song]

the VOA associated to (A_{k-1}, A_{n-1}) with $\gcd(k,n)=1$ is a W-algebra minimal model [Buican, Nishinaka], [Cordova, Shao]

we are interested in the W_3-algebra, which has been studied in [Agarwal, Lee, Song]

Level 5

$$\left(W_{-3}L_{-2} - \frac{10}{7}W_{-5}\right)\left|\Omega, c=-\frac{114}{7}\right\rangle \tag{B.3}$$

strong

generators

Level 6

$$\left(\frac{102}{49}L_{-6} - \frac{24}{7}L_{-4}L_{-2} - \frac{6}{7}(L_{-3})^2 + (L_{-2})^3 - \frac{39}{7}(W_{-3})^2\right)\left|\Omega, c=-\frac{114}{7}\right\rangle \tag{B.4}$$

Example: The (A_2, A_3) Argyres--Douglas SCFT

[Kang, CL, Song]

the VOA associated to (A_{k-1}, A_{n-1}) with $\gcd(k,n) = 1$ is a W-algebra minimal model [Buican, Nishinaka], [Cordova, Shao]

we are interested in the W_3-algebra, which has been studied in [Agarwal, Lee, Song]

Level 5

$$\left(W_{-3}L_{-2} - \frac{10}{7}W_{-5}\right)\left|\Omega, c = -\frac{114}{7}\right\rangle \tag{B.3}$$

strong generators

Level 6

$$\left(\frac{102}{49}L_{-6} - \frac{24}{7}L_{-4}L_{-2} - \frac{6}{7}(L_{-3})^2 + (L_{-2})^3 - \frac{39}{7}(W_{-3})^2\right)\left|\Omega, c = -\frac{114}{7}\right\rangle \tag{B.4}$$

let $S = \mathbb{C}[x,y]$ be the bigraded ring with gradings given by

$\deg(x) = (3,2)$ and $\deg(y) = (2,1)$

consider the (inhomogeneous) ideal $I = (x^2 + y^3, xy)$

Example: The (A_2, A_3) Argyres--Douglas SCFT

[Kang, CL, Song]

let $S = \mathbb{C}[x, y]$ be the bigraded ring with gradings given by

$\deg(x) = (3,2)$ and $\deg(y) = (2,1)$

consider the (inhomogeneous) ideal $I = (x^2 + y^3, xy)$

the bifiltration that we want on R is that induced from the bigrading on $\mathbb{C}[x, y]$
after quotienting by the inhomogeneous ideal

Example: The (A_2, A_3) Argyres--Douglas SCFT

[Kang, CL, Song]

let $S = \mathbb{C}[x, y]$ be the bigraded ring with gradings given by

$\deg(x) = (3,2)$ and $\deg(y) = (2,1)$

consider the (inhomogeneous) ideal $I = (x^2 + y^3, xy)$

the bifiltration that we want on R is that induced from the bigrading on $\mathbb{C}[x, y]$
after quotienting by the inhomogeneous ideal

the nth jet scheme is $$J_n(R) = \mathbb{C}[x_0, y_0, \cdots, x_n, y_n]/I_n$$

where the ideal is generated by $$\sum_{\substack{i,j \geq 0 \\ i+j=m}} x_i x_j - \sum_{\substack{i,j,k \geq 0 \\ i+j+k=m}} y_i y_j y_k \,, \qquad \sum_{\substack{i,j \geq 0 \\ i+j=m}} x_i y_j \qquad (m = 0, \cdots, n)$$

Example: The (A_2, A_3) Argyres--Douglas SCFT

[Kang, CL, Song]

let $S = \mathbb{C}[x,y]$ be the bigraded ring with gradings given by

$\deg(x) = (3,2)$ and $\deg(y) = (2,1)$

consider the (inhomogeneous) ideal $I = (x^2 + y^3, xy)$

the bifiltration that we want on R is that induced from the bigrading on $\mathbb{C}[x,y]$
after quotienting by the inhomogeneous ideal

the nth jet scheme is $$J_n(R) = \mathbb{C}[x_0, y_0, \cdots, x_n, y_n]/I_n$$

where the ideal is generated by $$\sum_{\substack{i,j\geq 0 \\ i+j=m}} x_i x_j - \sum_{\substack{i,j,k\geq 0 \\ i+j+k=m}} y_i y_j y_k \,, \qquad \sum_{\substack{i,j\geq 0 \\ i+j=m}} x_i y_j \qquad (m = 0, \cdots, n)$$

to determine the Hilbert series up to q^{13} we study $J_{11}(R)$, which has 24 polynomials in I_{11}

$\rightarrow$ we use Macaulay2 to find a Groebner basis

Example: The (A_2, A_3) Argyres--Douglas SCFT

[Kang, CL, Song]

let $S = \mathbb{C}[x, y]$ be the bigraded ring with gradings given by

$\deg(x) = (3,2)$ and $\deg(y) = (2,1)$

consider the (inhomogeneous) ideal $I = (x^2 + y^3, xy)$

the bifiltration that we want on R is that induced from the bigrading on $\mathbb{C}[x, y]$
after quotienting by the inhomogeneous ideal

$$
\begin{aligned}
\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_\infty(R))) &= 1 + q^2 T + q^3(T^2 + T) \\
&+ q^4(2T^2 + T) + q^5(2T^2 + T) \\
&+ q^6(2T^3 + 3T^2 + T) + q^7(3T^3 + 3T^2 + T) \\
&+ q^8(T^4 + 5T^3 + 4T^2 + T) \\
&+ q^9(2T^4 + 7T^3 + 4T^2 + T) \\
&+ q^{10}(5T^4 + 9T^3 + 5T^2 + T) \\
&+ q^{11}(8T^4 + 11T^4 + 5T^2 + T) \\
&+ q^{12}(2T^5 + 13T^4 + 14T^3 + 6T^2 + T) \\
&+ q^{13}(4T^5 + 17T^4 + 16T^3 + 6T^2 + T) + \cdots .
\end{aligned}
$$

$= I_{\mathrm{Mac}}(q, T)$ [Agarwal, Lee, Song]

Example: The $(A_2, A_{N>3})$ Argyres--Douglas SCFT

[Kang, CL, Song]

the null relations in the relevant W_3-algebra are also known [Agarwal, Lee, Song]

Example: The $(A_2, A_{N>3})$ Argyres--Douglas SCFT

[Kang, CL, Song]

the null relations in the relevant W_3-algebra are also known [Agarwal, Lee, Song]

Level 6

$$\left(\frac{102}{49}L_{-6} - \frac{24}{7}L_{-4}L_{-2} - \frac{6}{7}(L_{-3})^2 + (L_{-2})^3 - \frac{39}{7}(W_{-3})^2\right)\left|\Omega, c = -\frac{114}{7}\right\rangle \quad \text{(B.4)}$$

$$\left(\frac{27}{4}L_{-6} - \frac{21}{4}L_{-4}L_{-2} - \frac{51}{32}(L_{-3})^2 + (L_{-2})^3 - \frac{279}{16}(W_{-3})^2\right)|\Omega, c = -23\rangle \quad \text{(B.5)}$$

Level 7

$$\left(-\frac{5}{8}W_{-7} - \frac{3}{8}L_{-4}W_{-3} - \frac{11}{4}W_{-5}L_{-2} - \frac{3}{16}W_{-4}L_{-3} + W_{-3}L_{-2}L_{-2}\right)|\Omega, c = -23\rangle \quad \text{(B.6)}$$

$$\left(L_{-3}L_{-2}^2 - \frac{9}{4}L_{-5}L_{-2} + \frac{27L_{-7}}{8} - \frac{15}{8}L_{-4}L_{-3} - \frac{93}{8}W_{-4}W_{-3}\right)|\Omega, c = -23\rangle \quad \text{(B.7)}$$

Level 8

$$\begin{aligned}&\left(\frac{1413}{5}L_{-8} - 15L_{-6}L_{-2} - \frac{531}{20}L_{-5}L_{-3} - \frac{1011}{100}(L_{-4})^2 + \frac{1599}{10}W_{-5}W_{-3} + \frac{369}{40}(W_{-4})^2\right.\\ &\left. - \frac{48}{5}L_{-4}(L_{-2})^2 - \frac{15}{4}(L_{-3})^2L_{-2} + (L_{-2})^4 - \frac{123}{2}(W_{-3})^2L_{-2}\right)\left|\Omega, c = -\frac{186}{5}\right\rangle\end{aligned} \quad \text{(B.8)}$$

Level 9

$$\begin{aligned}&\left(-\frac{123}{8}L_{-3}W_{-3}^2 - \frac{123}{4}L_{-2}W_{-4}W_{-3} - \frac{1}{16}15L_{-3}^3 + L_{-2}^3L_{-3}\right.\\ &- \frac{141}{5}L_{-6}L_{-3} - \frac{111}{20}L_{-4}L_{-2}L_{-3} - \frac{27}{10}L_{-5}L_{-2}^2 + \frac{11061L_{-9}}{25}\\ &\left.- \frac{1587}{50}L_{-5}L_{-4} - \frac{309}{10}L_{-7}L_{-2} + \frac{246}{5}W_{-5}W_{-4} + \frac{4797}{40}W_{-6}W_{-3}\right)\left|\Omega, c = -\frac{186}{5}\right\rangle\end{aligned} \quad \text{(B.9)}$$

$$\begin{aligned}&\left(L_{-2}^3W_{-3} - \frac{42}{11}L_{-2}^2W_{-5} - \frac{877}{121}L_{-2}W_{-7} - \frac{9}{11}L_{-3}L_{-2}W_{-4}\right.\\ &- \frac{51}{11}L_{-4}L_{-2}W_{-3} + \frac{777}{121}L_{-3}W_{-6} + \frac{596}{121}L_{-4}W_{-5} + \frac{38}{121}L_{-5}W_{-4}\\ &\left.- \frac{15}{11}L_{-3}^2W_{-3} + \frac{948}{121}L_{-6}W_{-3} + \frac{27620}{1331}W_{-9} - \frac{138}{11}W_{-3}^3\right)\left|\Omega, c = -\frac{490}{11}\right\rangle\end{aligned} \quad \text{(B.10)}$$

Example: The $(A_2, A_{N>3})$ Argyres--Douglas SCFT

[Kang, CL, Song]

the null relations in the relevant W_3-algebra are also known [Agarwal, Lee, Song]

Level 6

$$\left(\frac{102}{49}L_{-6} - \frac{24}{7}L_{-4}L_{-2} - \frac{6}{7}(L_{-3})^2 + (L_{-2})^3 - \frac{39}{7}(W_{-3})^2\right)\left|\Omega, c = -\frac{114}{7}\right\rangle \tag{B.4}$$

$$\left(\frac{27}{4}L_{-6} - \frac{21}{4}L_{-4}L_{-2} - \frac{51}{32}(L_{-3})^2 + (L_{-2})^3 - \frac{279}{16}(W_{-3})^2\right)|\Omega, c = -23\rangle \tag{B.5}$$

Level 7

$$\left(-\frac{5}{8}W_{-7} - \frac{3}{8}L_{-4}W_{-3} - \frac{11}{4}W_{-5}L_{-2} - \frac{3}{16}W_{-4}L_{-3} + W_{-3}L_{-2}L_{-2}\right)|\Omega, c = -23\rangle \tag{B.6}$$

$$\left(L_{-3}L^2_{-2} - \frac{9}{4}L_{-5}L_{-2} + \frac{27L_{-7}}{8} - \frac{15}{8}L_{-4}L_{-3} - \frac{93}{8}W_{-4}W_{-3}\right)|\Omega, c = -23\rangle \tag{B.7}$$

Level 8

$$\begin{aligned}&\left(\frac{1413}{5}L_{-8} - 15L_{-6}L_{-2} - \frac{531}{20}L_{-5}L_{-3} - \frac{1011}{100}(L_{-4})^2 + \frac{1599}{10}W_{-5}W_{-3} + \frac{369}{40}(W_{-4})^2\right.\\&\left.\quad - \frac{48}{5}L_{-4}(L_{-2})^2 - \frac{15}{4}(L_{-3})^2L_{-2} + (L_{-2})^4 - \frac{123}{2}(W_{-3})^2L_{-2}\right)\left|\Omega, c = -\frac{186}{5}\right\rangle\end{aligned} \tag{B.8}$$

Level 9

$$\begin{aligned}&\left(-\frac{123}{8}L_{-3}W^2_{-3} - \frac{123}{4}L_{-2}W_{-4}W_{-3} - \frac{1}{16}15L^3_{-3} + L^3_{-2}L_{-3}\right.\\&\quad - \frac{141}{5}L_{-6}L_{-3} - \frac{111}{20}L_{-4}L_{-2}L_{-3} - \frac{27}{10}L_{-5}L^2_{-2} + \frac{11061L_{-9}}{25}\\&\left.\quad - \frac{1587}{50}L_{-5}L_{-4} - \frac{309}{10}L_{-7}L_{-2} + \frac{246}{5}W_{-5}W_{-4} + \frac{4797}{40}W_{-6}W_{-3}\right)\left|\Omega, c = -\frac{186}{5}\right\rangle\end{aligned} \tag{B.9}$$

$$\begin{aligned}&\left(L^3_{-2}W_{-3} - \frac{42}{11}L^2_{-2}W_{-5} - \frac{877}{121}L_{-2}W_{-7} - \frac{9}{11}L_{-3}L_{-2}W_{-4}\right.\\&\quad - \frac{51}{11}L_{-4}L_{-2}W_{-3} + \frac{777}{121}L_{-3}W_{-6} + \frac{596}{121}L_{-4}W_{-5} + \frac{38}{121}L_{-5}W_{-4}\\&\left.\quad - \frac{15}{11}L^2_{-3}W_{-3} + \frac{948}{121}L_{-6}W_{-3} + \frac{27620}{1331}W_{-9} - \frac{138}{11}W^3_{-3}\right)\left|\Omega, c = -\frac{490}{11}\right\rangle\end{aligned} \tag{B.10}$$

thus we can derive the rings!

Example: The $(A_2, A_{N>3})$ Argyres--Douglas SCFT

[Kang, CL, Song]

the null relations in the relevant W_3-algebra are also known [Agarwal, Lee, Song]

$$\begin{aligned}
(A_2, A_4) &\quad : \quad R = S/(xy^2, x^2 + y^3)\,, \\
(A_2, A_6) &\quad : \quad R = S/(x^3 + 4xy^3, y^4 + 3x^2 y)\,, \\
(A_2, A_7) &\quad : \quad R = S/(y^5 + 5y^2x^2, 2x^3 + 3xy^3)
\end{aligned}$$

thus we can derive the rings!

Example: The $(A_2, A_{N>3})$ Argyres--Douglas SCFT

[Kang, CL, Song]

$$
\begin{aligned}
(A_2, A_4) &\quad : \quad R = S/(xy^2, x^2+y^3)\,, \\
(A_2, A_6) &\quad : \quad R = S/(x^3+4xy^3, y^4+3x^2y)\,, \\
(A_2, A_7) &\quad : \quad R = S/(y^5+5y^2x^2, 2x^3+3xy^3)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_\infty(R))) &= 1 + q^2T + q^3(T^2+T) \\
&+ q^4(2T^2+T) + q^5(T^3+2T^2+T) \\
&+ q^6(3T^3+3T^2+T) + q^7(4T^3+3T^2+T) \\
&+ q^8(3T^4+6T^3+4T^2+T) \\
&+ q^9(5T^4+8T^3+4T^2+T) \\
&+ q^{10}(T^5+9T^4+10T^3+5T^2+T) \\
&+ q^{11}(3T^5+13T^4+12T^4+5T^2+T) \\
&+ q^{12}(8T^5+19T^4+15T^3+6T^2+T) \\
&+ q^{13}(14T^5+24T^4+17T^3+6T^2+T) + \cdots .
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_\infty(R))) &= 1 + q^2T + q^3(T^2+T) \\
&+ q^4(2T^2+T) + q^5(T^3+2T^2+T) \\
&+ q^6(T^4+3T^3+3T^2+T) \\
&+ q^7(2T^4+4T^3+3T^2+T) \\
&+ q^8(5T^4+6T^3+4T^2+T) \\
&+ q^9(2T^5+7T^4+8T^3+4T^2+T) \\
&+ q^{10}(6T^5+11T^4+10T^3+5T^2+T) \\
&+ q^{11}(T^6+10T^5+15T^4+12T^4+5T^2+T) \\
&+ q^{12}(6T^6+17T^5+21T^4+15T^3+6T^2+T) \\
&+ q^{13}(11T^6+25T^5+26T^4+17T^3+6T^2+T) \\
&+ q^{14}(2T^7+21T^6+36T^5 \\
&\qquad +34T^4+20T^3+7T^2+T) \\
&+ q^{15}(7T^7+34T^6+49T^5 \\
&\qquad +41T^4+23T^3+7T^2+T) + \cdots .
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_\infty(R))) &= 1 + q^2T + q^3(T^2+T) \\
&+ q^4(2T^2+T) + q^5(T^3+2T^2+T) \\
&+ q^6(T^4+3T^3+3T^2+T) \\
&+ q^7(2T^4+4T^3+3T^2+T) \\
&+ q^8(T^5+5T^4+6T^3+4T^2+T) \\
&+ q^9(3T^5+7T^4+8T^3+4T^2+T) \\
&+ q^{10}(7T^5+11T^4+10T^3+5T^2+T) \\
&+ q^{11}(3T^6+11T^5+15T^4+12T^4+5T^2+T) \\
&+ q^{12}(9T^6+18T^5+21T^4+15T^3+6T^2+T) \\
&+ q^{13}(T^7+15T^6+26T^5+26T^4 \\
&\qquad +17T^3+6T^2+T) \\
&+ q^{14}(7T^7+26T^6+37T^5+34T^4 \\
&\qquad +20T^3+7T^2+T) \\
&+ q^{15}(15T^7+40T^6+50T^5+41T^4 \\
&\qquad +23T^3+7T^2+T) \\
&+ q^{16}(3T^8+29T^7+59T^6+66T^5 \\
&\qquad +51T^4+26T^3+8T^2+T) + \cdots .
\end{aligned}
$$

matches with the Macdonald indices as worked out in [Agarwal, Lee, Song]

(A_{k-1}, A_{N-1}) and Jacobian Rings

[Kang, CL, Song]

when gcd(k, N) = 1, all known bifiltered rings are Jacobian rings:

$$S = \mathbb{C}[x_1, \cdots, x_{k-1}], \quad \deg(x_i) = (i+1, i)$$

$$R = S / (\partial_{x_1} P^{N+k+1}, \cdots, \partial_{x_{k-1}} P^{N+k+1})$$

degree $N+k+1$ (in first degree) polynomial in the x_i

(A_{k-1}, A_{N-1}) and Jacobian Rings

[Kang, CL, Song]

when gcd(k, N) = 1, all known bifiltered rings are Jacobian rings:

$$S = \mathbb{C}[x_1, \cdots, x_{k-1}]\,, \quad \deg(x_i) = (i+1, i)$$

$$R = S / (\partial_{x_1} P^{N+k+1}, \cdots, \partial_{x_{k-1}} P^{N+k+1})$$

degree $N+k+1$ (in first degree) polynomial in the x_i

where the null relations in the W-algebra are not known

we can find the polynomial P^{N+k+1} via the Macdonald index

this is a *prediction* for the OPE decoupling/null relations

(A_{k-1}, A_{N-1}) and Jacobian Rings

[Kang, CL, Song]

when gcd(k, N) = 1, all known bifiltered rings are Jacobian rings:

$$S = \mathbb{C}[x_1, \cdots, x_{k-1}]\,, \quad \deg(x_i) = (i+1, i)$$

$$R = S\,/\,(\partial_{x_1} P^{N+k+1}, \cdots, \partial_{x_{k-1}} P^{N+k+1})$$

degree $N+k+1$ (in first degree) polynomial in the x_i

where the null relations in the W-algebra are not known

we can find the polynomial P^{N+k+1} via the Macdonald index

this is a *prediction* for the OPE decoupling/null relations

$$\begin{aligned}
(A_2, A_1) &: \quad P^6(x,y) = x^3 + y^2\,,\\
(A_2, A_3) &: \quad P^8(x,y) = y^4 + x^2 y\,,\\
(A_2, A_4) &: \quad P^9(x,y) = x^3 + x y^3\,,\\
(A_2, A_6) &: \quad P^{11}(x,y) = x^3 y + x y^4\,,\\
(A_2, A_7) &: \quad P^{12}(x,y) = y^6 + \sqrt{30}\, x^2 y^3 + x^4\,,\\
(A_2, A_9) &: \quad P^{14}(x,y) = y^7 + \sqrt{21}\, x^2 y^4 + x^4 y\,,\\
(A_2, A_{10}) &: \quad P^{15}(x,y) = x^5 + \frac{10}{\sqrt{6}}\, x^3 y^3 + x y^6\,.
\end{aligned}$$

On the Principle of Syzygy Maximization

[Kang, CL, Song]

On the Principle of Syzygy Maximization [Kang, CL, Song]

consider the (A_2, A_9) AD theory

W-algebra known to have: strong generators, L_{-2}, W_{-3} of degrees 2, 3

null relations of degrees 11, 12 (but the relations are not known!)

On the Principle of Syzygy Maximization [Kang, CL, Song]

consider the (A_2, A_9) AD theory

W-algebra known to have: strong generators, L_{-2}, W_{-3} of degrees 2, 3

null relations of degrees 11, 12 (but the relations are not known!)

assuming Jacobian: $P(x_2, x_3) = \beta x_2^7 + x_3^2 x_2^4 + x_3^4 x_2$

after rescaling/normalization,
free parameter

On the Principle of Syzygy Maximization

[Kang, CL, Song]

consider the (A_2, A_9) AD theory

W-algebra known to have: strong generators, L_{-2}, W_{-3} of degrees 2, 3

null relations of degrees 11, 12 (but the relations are not known!)

assuming Jacobian: $P(x_2, x_3) = \beta x_2^7 + x_3^2 x_2^4 + x_3^4 x_2$

after rescaling/normalization, free parameter

for generic β the ring $R = \mathbb{C}[x_2, x_3]/(\partial_{x_2} P, \partial_{x_3} P)$ does not reproduce the Schur limit of the Macdonald index via algebro-geometric bootstrapping

On the Principle of Syzygy Maximization

[Kang, CL, Song]

consider the (A_2, A_9) AD theory

assuming Jacobian: $P(x_2, x_3) = \beta x_2^7 + x_3^2 x_2^4 + x_3^4 x_2$

a syzygy of (q, p)-degree (k + N + 2, 1) is a relation-between-relations in the first jet scheme

$$J_1(R) = \mathbb{C}[x_{2,0}, x_{3,0}, x_{2,1}, x_{3,1},] / (f_{2,0}, f_{3,0}, f_{2,1}, f_{3,1})$$

$f_{d,p}$ = coefficient of t^p in jet expansion of $\partial_{x_d} P$

On the Principle of Syzygy Maximization [Kang, CL, Song]

consider the (A_2, A_9) AD theory

assuming Jacobian: $P(x_2, x_3) = \beta x_2^7 + x_3^2 x_2^4 + x_3^4 x_2$

a syzygy of (q, p)-degree (k + N + 2, 1) is a relation-between-relations in the first jet scheme

$$J_1(R) = \mathbb{C}[x_{2,0}, x_{3,0}, x_{2,1}, x_{3,1},] / (f_{2,0}, f_{3,0}, f_{2,1}, f_{3,1})$$

$f_{d,p}$ = coefficient of t^p in jet expansion of $\partial_{x_d} P$

$$\alpha_1 x_{2,0} x_{2,1} f_{3,0} + \alpha_2 x_{2,0}^2 f_{3,1} + \alpha_3 x_{3,1} f_{2,0} + \alpha_4 x_{3,0} f_{2,1} \stackrel{!}{=} 0$$

with non-zero α_i

On the Principle of Syzygy Maximization

[Kang, CL, Song]

consider the (A_2, A_9) AD theory

assuming Jacobian: $P(x_2, x_3) = \beta x_2^7 + x_3^2 x_2^4 + x_3^4 x_2$

a syzygy of (q, p)-degree (k + N + 2, 1) is a relation-between-relations in the first jet scheme

$$J_1(R) = \mathbb{C}[x_{2,0}, x_{3,0}, x_{2,1}, x_{3,1},] / (f_{2,0}, f_{3,0}, f_{2,1}, f_{3,1})$$

$f_{d,p}$ = coefficient of t^p in jet expansion of $\partial_{x_d} P$

$$\alpha_1 x_{2,0} x_{2,1} f_{3,0} + \alpha_2 x_{2,0}^2 f_{3,1} + \alpha_3 x_{3,1} f_{2,0} + \alpha_4 x_{3,0} f_{2,1} \stackrel{!}{=} 0$$

with non-zero α_i $\Rightarrow \beta = \sqrt{21}$

On the Principle of Syzygy Maximization [Kang, CL, Song]

in fact, matching with the closed form expression of the Schur limit of the Macdonald index

$$I_{(A_{k-1},A_{N-1})}(q) = \mathrm{PE}\left[\frac{q^2+q^3+\cdots+q^k-q^{N+1}-q^{N+2}-\cdots-q^{N+k-1}}{(1-q)(1-q^{k+N})}\right]$$

is only possible if

$$R \text{ is a complete intersection},$$

$$\beta_{2,(k+N+1,1)}(J_1R) = \beta_{2,(k+N+2,1)}(J_1R) = 1$$

Betti number $\beta_{2,(q,p)}$ counts the number of minimal syzygies of bidegree (q, p)

On the Principle of Syzygy Maximization

[Kang, CL, Song]

in fact, matching with the closed form expression of the Schur limit of the Macdonald index

$$I_{(A_{k-1},A_{N-1})}(q) = \mathrm{PE}\left[\frac{q^2+q^3+\cdots+q^k-q^{N+1}-q^{N+2}-\cdots-q^{N+k-1}}{(1-q)(1-q^{k+N})}\right]$$

is only possible if

$$R \text{ is a complete intersection}\,,$$

$$\beta_{2,(k+N+1,1)}(J_1R)=\beta_{2,(k+N+2,1)}(J_1R)=1$$

⇒ Jacobian ring ⇒ remaining coefficients

surprising: this uniquely fixes all coefficients

On the Principle of Syzygy Maximization

[Kang, CL, Song]

in fact, matching with the closed form expression of the Schur limit of the Macdonald index

$$I_{(A_{k-1},A_{N-1})}(q) = \mathrm{PE}\left[\frac{q^2+q^3+\cdots+q^k-q^{N+1}-q^{N+2}-\cdots-q^{N+k-1}}{(1-q)(1-q^{k+N})}\right]$$

is only possible if

$$R \text{ is a complete intersection}\,,$$

$$\beta_{2,(k+N+1,1)}(J_1R) = \beta_{2,(k+N+2,1)}(J_1R) = 1$$

⇒ Jacobian ring ⇒ remaining coefficients

surprising: this uniquely fixes all coefficients

imposing a geometric extremization condition on syzygies of the first(!) jet scheme is sufficient to fix R uniquely and recovers the Schur index of (A_{k-1},A_{N-1})

On the Principle of Syzygy Maximization

[Kang, CL, Song]

Theory	Polynomial $P(x_2, x_3, \cdots)$
(A_2, A_3)	$x_2^4 + x_2 x_3^2$
(A_2, A_4)	$x_2^3 x_3 + x_3^3$
(A_2, A_6)	$x_2^4 x_3 + x_2 x_3^3$
(A_2, A_7)	$\frac{1}{30} x_2^6 + x_2^3 x_3^2 + x_3^4$
(A_2, A_9)	$\frac{1}{21} x_2^7 + x_2^4 x_3^2 + x_2 x_3^4$
(A_2, A_{10})	$x_2^6 x_3 + x_2^3 x_3^3 + \frac{3}{50} x_3^5$
(A_2, A_{12})	$\frac{3}{35} x_2^7 x_3 + x_2^4 x_3^3 + x_2 x_3^5$
(A_2, A_{13})	$\frac{5}{72} x_2^9 + x_2^6 x_3^2 + x_2^3 x_3^4 + \frac{2}{25} x_3^6$
(A_2, A_{15})	$\frac{8}{7875} x_2^{10} + \frac{4}{35} x_2^7 x_3^2 + x_2^4 x_3^4 + x_2 x_3^6$
(A_2, A_{16})	$\frac{1}{8} x_2^9 x_3 + x_2^6 x_3^3 + x_2^3 x_3^5 + \frac{2}{21} x_3^7$
(A_2, A_{18})	$\frac{8}{3087} x_2^{10} x_3 + \frac{20}{147} x_2^7 x_3^3 + x_2^4 x_3^5 + x_2 x_3^7$
(A_2, A_{19})	$\frac{1}{396} x_2^{12} + \frac{1}{6} x_2^9 x_3^2 + x_2^6 x_3^4 + x_2^3 x_3^6 + \frac{3}{28} x_3^8$
(A_2, A_{21})	$\frac{3}{249704} x_2^{13} + \frac{3}{686} x_2^{10} x_3^2 + \frac{15}{98} x_2^7 x_3^4 + x_2^4 x_3^6 + x_2 x_3^8$
(A_2, A_{22})	$\frac{125}{19404} x_2^{12} x_3 + \frac{25}{126} x_2^9 x_3^3 + x_2^6 x_3^5 + x_2^3 x_3^7 + \frac{7}{60} x_3^9$
(A_3, A_4)	$\frac{1}{10} x_2^5 + x_2^2 x_3^2 + x_2 x_4^2 + x_3^2 x_4$
(A_3, A_6)	$\frac{9}{2} x_2^4 x_4 + x_2^3 x_3^2 + x_2 x_3^2 x_4 + \frac{1}{108} x_3^4 + x_4^3$
(A_3, A_8)	$\frac{x_2^5 x_4}{120} + \frac{x_2^4 x_3^2}{72} + \frac{x_2^2 x_3^2 x_4}{3} + \frac{x_2 x_3^4}{18} + x_2 x_4^3 + x_3^2 x_4^2$
(A_3, A_{10})	$\frac{x_2^8}{47029248} + \frac{x_2^5 x_3^2}{7776} + \frac{x_2^4 x_4^2}{1296} + \frac{x_2^3 x_3^2 x_4}{54} + \frac{x_2^2 x_3^4}{18}$ $+ x_2 x_3^2 x_4^2 + x_3^4 x_4 + x_4^4$
(A_4, A_5)	$\frac{61776859}{5} x_2^6 + 184899 x_2^4 x_4 + 10368 x_2^3 x_3^2 + x_2^2 x_3 x_5$ $+ 861 x_2^2 x_4^2 + x_2 x_3^2 x_4 + x_2 x_5^2 + x_3^4 + x_3 x_4 x_5 + x_4^3$

for each theory, a single geometrically determined polynomial is sufficient to determine (all?) information about the 1/4-BPS operators

Examples: Higgs branch $\neq$ point

Example: The (A_1, D_3) Argyres--Douglas SCFT

[Kang, CL, Song]

the (A_1, D_{2n+1}) theories have recently been studied in [Andrews, Banerjee, Bhargava, Singh, Tao]

$$R = \mathbb{C}[x, y, z, w]/(w - (xz + y^2), wx, wy, wz)$$

$$\deg(x) = \deg(y) = \deg(z) = (1,1)\,, \quad \deg(w) = (2,1)$$

Example: The (A_1, D_3) Argyres--Douglas SCFT

[Kang, CL, Song]

the (A_1, D_{2n+1}) theories have recently been studied in [Andrews, Banerjee, Bhargava, Singh, Tao]

$$R = \mathbb{C}[x, y, z, w]/(w - (xz + y^2), wx, wy, wz)$$

$$\deg(x) = \deg(y) = \deg(z) = (1,1)\,, \quad \deg(w) = (2,1)$$

the bifiltration that we want on R is that induced from the bigrading on $\mathbb{C}[x, y, z, w]$ after quotienting by the inhomogeneous ideal

Example: The (A_1, D_3) Argyres--Douglas SCFT

[Kang, CL, Song]

the (A_1, D_{2n+1}) theories have recently been studied in [Andrews, Banerjee, Bhargava, Singh, Tao]

$$R = \mathbb{C}[x, y, z, w]/(w - (xz + y^2), wx, wy, wz)$$
$$\deg(x) = \deg(y) = \deg(z) = (1,1)\,, \quad \deg(w) = (2,1)$$

the bifiltration that we want on R is that induced from the bigrading on $\mathbb{C}[x, y, z, w]$ after quotienting by the inhomogeneous ideal

the Higgs branch of (A_1, D_3) is $\mathbb{C}^2/\mathbb{Z}_2$ which has an $\mathfrak{su}(2)$ isometry $\Rightarrow$ ideal is homogeneous under an additional grading

$$\deg(x) = 1\,,\ \deg(z) = -1\,,\ \deg(y) = \deg(w) = 0$$

Example: The (A_1, D_3) Argyres--Douglas SCFT

[Kang, CL, Song]

the (A_1, D_{2n+1}) theories have recently been studied in [Andrews, Banerjee, Bhargava, Singh, Tao]

$$R = \mathbb{C}[x,y,z,w]/(w-(xz+y^2), wx, wy, wz)$$
$$\deg(x) = \deg(y) = \deg(z) = (1,1)\,, \quad \deg(w) = (2,1)$$

the bifiltration that we want on R is that induced from the bigrading on $\mathbb{C}[x,y,z,w]$ after quotienting by the inhomogeneous ideal

the Higgs branch of (A_1, D_3) is $\mathbb{C}^2/\mathbb{Z}_2$ which has an $\mathfrak{su}(2)$ isometry $\Rightarrow$ ideal is homogeneous under an additional grading

$$\deg(x) = 1\,, \ \deg(z) = -1\,, \ \deg(y) = \deg(w) = 0$$

$$\begin{aligned}
\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_\infty(R))) = 1 &+ \chi_{\mathbf{3}}(a)qT \\
&+ q^2(\chi_{\mathbf{5}}(a)T^2 + \chi_{\mathbf{3}\oplus\mathbf{1}}(a)T) \\
&+ q^3(\chi_{\mathbf{7}}(a)T^3 + \chi_{\mathbf{5}\oplus\mathbf{3}}(a)T^2 + \chi_{\mathbf{3}\oplus\mathbf{1}}(a)T) \\
&+ q^4(\chi_{\mathbf{9}}(a)T^4 + \chi_{\mathbf{7}\oplus\mathbf{5}}(a)T^3 \\
&\quad + \chi_{2\cdot\mathbf{5}\oplus 2\cdot\mathbf{3}\oplus\mathbf{1}}(a)T^2 + \chi_{\mathbf{3}\oplus\mathbf{1}}(a)T) \\
&+ q^5(\chi_{\mathbf{11}}(a)T^5 + \chi_{\mathbf{9}\oplus\mathbf{7}}(a)T^4 + \chi_{2\cdot\mathbf{7}\oplus 2\cdot\mathbf{5}\oplus\mathbf{3}}(a)T^3 \\
&\quad + \chi_{2\cdot\mathbf{5}\oplus 4\cdot\mathbf{3}\oplus\mathbf{1}}(a)T^2 + \chi_{\mathbf{3}\oplus\mathbf{1}}(a)T) \\
&+ q^6(\chi_{\mathbf{13}}(a)T^6 + \chi_{\mathbf{11}\oplus\mathbf{9}}(a)T^5 \\
&\quad + \chi_{2\cdot\mathbf{9}\oplus 2\cdot\mathbf{7}\oplus\mathbf{5}}(a)T^4 + \chi_{3\cdot\mathbf{7}\oplus 4\cdot\mathbf{5}\oplus 2\cdot\mathbf{3}\oplus\mathbf{1}}(a)T^3 \\
&\quad + \chi_{3\cdot\mathbf{5}\oplus 5\cdot\mathbf{3}\oplus 3\cdot\mathbf{1}}(a)T^2 + \chi_{\mathbf{3}\oplus\mathbf{1}}(a)T) \\
&+ q^7(\chi_{\mathbf{15}}(a)T^7 + \chi_{\mathbf{13}\oplus\mathbf{11}}(a)T^6 \\
&\quad + \chi_{2\cdot\mathbf{11}\oplus 2\cdot\mathbf{9}\oplus\mathbf{7}}(a)T^5 + \chi_{3\cdot\mathbf{9}\oplus 4\cdot\mathbf{7}\oplus 2\cdot\mathbf{5}\oplus\mathbf{3}}(a)T^4 \\
&\quad + \chi_{4\cdot\mathbf{7}\oplus 7\cdot\mathbf{5}\oplus 5\cdot\mathbf{3}\oplus\mathbf{1}}(a)T^3 \\
&\quad + \chi_{3\cdot\mathbf{5}\oplus 7\cdot\mathbf{3}\oplus 3\cdot\mathbf{1}}(a)T^2 + \chi_{\mathbf{3}\oplus\mathbf{1}}(a)T) + \cdots .
\end{aligned}$$

Example: The (A_1, D_3) Argyres--Douglas SCFT

[Kang, CL, Song]

the (A_1, D_{2n+1}) theories have recently been studied in [Andrews, Banerjee, Bhargava, Singh, Tao]

$$R = \mathbb{C}[x, y, z, w]/(w - (xz + y^2), wx, wy, wz)$$
$$\deg(x) = \deg(y) = \deg(z) = (1,1)\,, \quad \deg(w) = (2,1)$$

the bifiltration that we want on R is that induced from the bigrading on $\mathbb{C}[x, y, z, w]$ after quotienting by the inhomogeneous ideal

the Higgs branch of (A_1, D_3) is $\mathbb{C}^2/\mathbb{Z}_2$ which has an $\mathfrak{su}(2)$ isometry $\Rightarrow$ ideal is homogeneous under an additional grading

$$\deg(x) = 1\,,\ \deg(z) = -1\,,\ \deg(y) = \deg(w) = 0$$

$$\begin{aligned}
\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_\infty(R))) &= 1 + \chi_{\mathbf{3}}(a)qT \\
&+ q^2(\chi_{\mathbf{5}}(a)T^2 + \chi_{\mathbf{3}\oplus\mathbf{1}}(a)T) \\
&+ q^3(\chi_{\mathbf{7}}(a)T^3 + \chi_{\mathbf{5}\oplus\mathbf{3}}(a)T^2 + \chi_{\mathbf{3}\oplus\mathbf{1}}(a)T) \\
&+ q^4(\chi_{\mathbf{9}}(a)T^4 + \chi_{\mathbf{7}\oplus\mathbf{5}}(a)T^3 \\
&\quad + \chi_{2\cdot\mathbf{5}\oplus 2\cdot\mathbf{3}\oplus\mathbf{1}}(a)T^2 + \chi_{\mathbf{3}\oplus\mathbf{1}}(a)T) \\
&+ q^5(\chi_{\mathbf{11}}(a)T^5 + \chi_{\mathbf{9}\oplus\mathbf{7}}(a)T^4 + \chi_{2\cdot\mathbf{7}\oplus 2\cdot\mathbf{5}\oplus\mathbf{3}}(a)T^3 \\
&\quad + \chi_{2\cdot\mathbf{5}\oplus 4\cdot\mathbf{3}\oplus\mathbf{1}}(a)T^2 + \chi_{\mathbf{3}\oplus\mathbf{1}}(a)T) \\
&+ q^6(\chi_{\mathbf{13}}(a)T^6 + \chi_{\mathbf{11}\oplus\mathbf{9}}(a)T^5 \\
&\quad + \chi_{2\cdot\mathbf{9}\oplus 2\cdot\mathbf{7}\oplus\mathbf{5}}(a)T^4 + \chi_{3\cdot\mathbf{7}\oplus 4\cdot\mathbf{5}\oplus 2\cdot\mathbf{3}\oplus\mathbf{1}}(a)T^3 \\
&\quad + \chi_{3\cdot\mathbf{5}\oplus 5\cdot\mathbf{3}\oplus 3\cdot\mathbf{1}}(a)T^2 + \chi_{\mathbf{3}\oplus\mathbf{1}}(a)T) \\
&+ q^7(\chi_{\mathbf{15}}(a)T^7 + \chi_{\mathbf{13}\oplus\mathbf{11}}(a)T^6 \\
&\quad + \chi_{2\cdot\mathbf{11}\oplus 2\cdot\mathbf{9}\oplus\mathbf{7}}(a)T^5 + \chi_{3\cdot\mathbf{9}\oplus 4\cdot\mathbf{7}\oplus 2\cdot\mathbf{5}\oplus\mathbf{3}}(a)T^4 \\
&\quad + \chi_{4\cdot\mathbf{7}\oplus 7\cdot\mathbf{5}\oplus 5\cdot\mathbf{3}\oplus\mathbf{1}}(a)T^3 \\
&\quad + \chi_{3\cdot\mathbf{5}\oplus 7\cdot\mathbf{3}\oplus 3\cdot\mathbf{1}}(a)T^2 + \chi_{\mathbf{3}\oplus\mathbf{1}}(a)T) + \cdots\,.
\end{aligned}$$

the flavor-refined Macdonald index for (A_1, D_3)! matches with [Buican, Nishinaka]

Example: The (A_1, D_3) Argyres--Douglas SCFT

[Kang, CL, Song]

the (A_1, D_{2n+1}) theories have recently been studied in [Andrews, Banerjee, Bhargava, Singh, Tao]

$$R = \mathbb{C}[x, y, z, w]/(w - (xz + y^2), wx, wy, wz)$$

$$\deg(x) = \deg(y) = \deg(z) = (1,1)\,, \quad \deg(w) = (2,1)$$

the bifiltration that we want on R is that induced from the bigrading on $\mathbb{C}[x, y, z, w]$ after quotienting by the inhomogeneous ideal

in fact: w contributes the stress-tensor to the index

a puzzle: here we have introduced a coordinate w which *does not* correspond to a strong generator of the VOA $\Rightarrow$ why?

Example: The (A_1, D_3) Argyres--Douglas SCFT

[Kang, CL, Song]

the (A_1, D_{2n+1}) theories have recently been studied in [Andrews, Banerjee, Bhargava, Singh, Tao]

$$R = \mathbb{C}[x, y, z, w]/(w - (xz + y^2), wx, wy, wz)$$

$$\deg(x) = \deg(y) = \deg(z) = (1,1)\,, \quad \deg(w) = (2,1)$$

the bifiltration that we want on R is that induced from the bigrading on $\mathbb{C}[x, y, z, w]$ after quotienting by the inhomogeneous ideal

in fact: w contributes the stress-tensor to the index

a puzzle: here we have introduced a coordinate w which *does not* correspond to a strong generator of the VOA $\Rightarrow$ why?

as *rings* R is isomorphic to $R' = \mathbb{C}[x, y, z]/((xz + y^2)x, (xz + y^2)y, (xz + y^2)z)$ and our bifiltration on R can be pushed through the isomorphism to define a bifiltration on R'

however this bifiltration is different from one induced by a bigrading on $\mathbb{C}[x, y, z]$

introducing w allows the bifiltration to be induced by a bigrading

Example: The (A_1, D_3) Argyres--Douglas SCFT

[Kang, CL, Song]

the (A_1, D_{2n+1}) theories have recently been studied in [Andrews, Banerjee, Bhargava, Singh, Tao]

$$R = \mathbb{C}[x, y, z, w]/(w - (xz + y^2), wx, wy, wz)$$

$$\deg(x) = \deg(y) = \deg(z) = (1{,}1)\,, \quad \deg(w) = (2{,}1)$$

the bifiltration that we want on R is that induced from the bigrading on $\mathbb{C}[x, y, z, w]$ after quotienting by the inhomogeneous ideal

in fact: w contributes the stress-tensor to the index

a puzzle: here we have introduced a coordinate w which *does not* correspond to a strong generator of the VOA $\Rightarrow$ why?

as *rings* R is isomorphic to $R' = \mathbb{C}[x, y, z]/((xz + y^2)x, (xz + y^2)y, (xz + y^2)z)$ and our bifiltration on R can be pushed through the isomorphism to define a bifiltration on R'

however this bifiltration is different from one induced by a bigrading on $\mathbb{C}[x, y, z]$

introducing w allows the bifiltration to be induced by a bigrading

? what about the bifiltration structure from the VOA perspective? ?

Example: The (A_1, A_5) Argyres--Douglas SCFT

[Kang, CL, Song]

$$R = \mathbb{C}[x, y, z, w]/(xy + z^3 - zw, wx, wy, (xy + z^3)z)$$

$$\deg(x) = \deg(y) = (3/2, 3/2)\,, \quad \deg(z) = (1,1)\,, \quad \deg(w) = (2,1)$$

the bifiltration is induced from the bigrading on $\mathbb{C}[x, y, z, w]$

Example: The (A_1, A_5) Argyres--Douglas SCFT

[Kang, CL, Song]

$$R = \mathbb{C}[x, y, z, w]/(xy + z^3 - zw, wx, wy, (xy + z^3)z)$$

$$\deg(x) = \deg(y) = (3/2, 3/2)\,, \quad \deg(z) = (1,1)\,, \quad \deg(w) = (2,1)$$

the bifiltration is induced from the bigrading on $\mathbb{C}[x, y, z, w]$

the Higgs branch is $\mathbb{C}^2/\mathbb{Z}_3 = \mathrm{Spec}(R)_{\mathrm{red}}$

Example: The (A_1, A_5) Argyres--Douglas SCFT

[Kang, CL, Song]

$$R = \mathbb{C}[x,y,z,w]/(xy+z^3-zw, wx, wy, (xy+z^3)z)$$

$$\deg(x)=\deg(y)=(3/2,3/2)\,,\quad \deg(z)=(1,1)\,,\quad \deg(w)=(2,1)$$

the bifiltration is induced from the bigrading on $\mathbb{C}[x,y,z,w]$

the Higgs branch is $\mathbb{C}^2/\mathbb{Z}_3 = \mathrm{Spec}(R)_{\mathrm{red}}$

$$\begin{aligned}\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_\infty(R))) &= 1+qT+2q^{\frac{3}{2}}T^{\frac{3}{2}}+q^2(T^2+2T)+q^{\frac{5}{2}}(2T^{\frac{5}{2}}+2T^{\frac{3}{2}})\\ &+q^3(3T^3+2T^2+2T)+q^{\frac{7}{2}}(2T^{\frac{7}{2}}+4T^{\frac{5}{2}}+2T^{\frac{3}{2}})+q^4(3T^4+4T^3+5T^2+2T)\\ &+q^{\frac{9}{2}}(4T^{\frac{9}{2}}+4T^{\frac{7}{2}}+8T^{\frac{5}{2}}+2T^{\frac{3}{2}})+q^5(3T^5+6T^4+9T^3+6T^2+2T)\\ &+q^{\frac{11}{2}}(4T^{\frac{11}{2}}+6T^{\frac{9}{2}}+10T^{\frac{7}{2}}+12T^{\frac{5}{2}}+2T^{\frac{3}{2}})+q^6(5T^6+6T^5+13T^4+14T^3+9T^2+2T)\\ &+q^{\frac{13}{2}}(4T^{\frac{13}{2}}+8T^{\frac{11}{2}}+14T^{\frac{9}{2}}+18T^{\frac{7}{2}}+16T^{\frac{5}{2}}+2T^{\frac{3}{2}})\\ &+q^7(5T^7+8T^6+15T^5+24T^4+22T^3+10T^2+2T)+\cdots\,.\end{aligned}$$

$= I_{\mathrm{Mac}}(q,T)$ [Buican, Nishinaka]

fun fact: R also comes from syzygy-maximization of a Jacobian ring

stay tuned: [Kang, CL, Song]

Summary

we proposed the existence of a "scheme-ification" of the Higgs branch of each 4d $\mathcal{N}=2$ SCFT which allows the recovery of information about protected operators in the Schur sector

Summary

we proposed the existence of a "scheme-ification" of the Higgs branch of each 4d $\mathcal{N}=2$ SCFT which allows the recovery of information about protected operators in the Schur sector

the key object is a *bifiltered affine scheme*

Summary

we proposed the existence of a "scheme-ification" of the Higgs branch of each 4d $\mathcal{N}=2$ SCFT which allows the recovery of information about protected operators in the Schur sector

the key object is a *bifiltered affine scheme*

we proposed bifiltered affine schemes for a variety of Argyres--Douglas theories, derived from decoupling phenomena in the OPEs of the 4d SCFT

Summary

we proposed the existence of a "scheme-ification" of the Higgs branch of each 4d $\mathcal{N}=2$ SCFT which allows the recovery of information about protected operators in the Schur sector

the key object is a *bifiltered affine scheme*

we proposed bifiltered affine schemes for a variety of Argyres--Douglas theories, derived from decoupling phenomena in the OPEs of the 4d SCFT

for (A_{k-1}, A_{N-1}) Argyres--Douglas theories, the bifiltered affine schemes follow from a geometric extremization principle: syzygy maximization!

Summary

we proposed the existence of a "scheme-ification" of the Higgs branch of each 4d $\mathcal{N}=2$ SCFT which allows the recovery of information about protected operators in the Schur sector

the key object is a *bifiltered affine scheme*

we proposed bifiltered affine schemes for a variety of Argyres--Douglas theories, derived from decoupling phenomena in the OPEs of the 4d SCFT

for (A_{k-1}, A_{N-1}) Argyres--Douglas theories, the bifiltered affine schemes follow from a geometric extremization principle: syzygy maximization!

Thank you!

How Many Jets Do We Need?

challenges: (technical) $J_\infty(R)$ has infinity-many generators and relations

How Many Jets Do We Need?

challenges: (technical) $J_\infty(R)$ has infinity-many generators and relations

suppose we have a ring $S = \mathbb{C}[x_1, \cdots, x_m]$ with $\deg(x_j) = (a_j, b_j)$ and $R = S/(f_1, \cdots, f_r)$ with $\deg(f_i) = (c_i, d_i)$

the lowest order contribution of $x_j^{(\alpha)}$ to $\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_\infty(R)))$ is $q^{a_j+\alpha} T^{b_j}$

the lowest order contribution of $f_i^{(\alpha)}$ to $\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_\infty(R)))$ is $q^{c_i+\alpha} T^{d_i}$

How Many Jets Do We Need?

challenges: (technical) $J_\infty(R)$ has infinity-many generators and relations

suppose we have a ring $S = \mathbb{C}[x_1, \cdots, x_m]$ with $\deg(x_j) = (a_j, b_j)$ and $R = S/(f_1, \cdots, f_r)$ with $\deg(f_i) = (c_i, d_i)$

the lowest order contribution of $x_j^{(\alpha)}$ to $\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_\infty(R)))$ is $q^{a_j+\alpha} T^{b_j}$

the lowest order contribution of $f_i^{(\alpha)}$ to $\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_\infty(R)))$ is $q^{c_i+\alpha} T^{d_i}$

$\Rightarrow$ $\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_\infty(R)))$ to order q^ℓ is determined by $\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_{\ell-d}(R)))$

d = min(a_j) -- the x_j with the smallest (first) degree in S

How Many Jets Do We Need?

challenges: (technical) $J_\infty(R)$ has infinity-many generators and relations

suppose we have a ring $S = \mathbb{C}[x_1, \cdots, x_m]$ with $\deg(x_j) = (a_j, b_j)$ and $R = S/(f_1, \cdots, f_r)$ with $\deg(f_i) = (c_i, d_i)$

the lowest order contribution of $x_j^{(\alpha)}$ to $\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_\infty(R)))$ is $q^{a_j+\alpha}T^{b_j}$

the lowest order contribution of $f_i^{(\alpha)}$ to $\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_\infty(R)))$ is $q^{c_i+\alpha}T^{d_i}$

$\Rightarrow$ $\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_\infty(R)))$ to order q^ℓ is determined by $\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_{\ell-d}(R)))$

d = min(a_j) -- the x_j with the smallest (first) degree in S

we can compute to arbitrary large order working over finitely-generated rings

Jet Schemes: An Explicit Example

$$R = \mathbb{C}[x, y]/(x^2 + y^3)$$

Jet Schemes: An Explicit Example

$$R = \mathbb{C}[x, y]/(x^2 + y^3)$$

$$J_1(R) = \mathbb{C}[x_0, x_1, y_0, y_1]/(x_0^2 + y_0^3,\ 2x_0x_1 + 3y_0^2y_1)$$

Jet Schemes: An Explicit Example

$$R = \mathbb{C}[x, y]/(x^2 + y^3)$$

$$J_1(R) = \mathbb{C}[x_0, x_1, y_0, y_1]/(x_0^2 + y_0^3,\ 2x_0x_1 + 3y_0^2y_1)$$

$$J_2(R) = \mathbb{C}[x_0, x_1, x_2, y_0, y_1, y_2]/(x_0^2 + y_0^3,\ 2x_0x_1 + 3y_0^2y_1,\ x_1^2 + 2x_2x_2 + 3y_0y_1^2 + 3y_0^2y_2)$$

Jet Schemes: An Explicit Example

$$R = \mathbb{C}[x, y]/(x^2 + y^3)$$

$$J_1(R) = \mathbb{C}[x_0, x_1, y_0, y_1]/(x_0^2 + y_0^3,\ 2x_0x_1 + 3y_0^2y_1)$$

$$J_2(R) = \mathbb{C}[x_0, x_1, x_2, y_0, y_1, y_2]/(x_0^2 + y_0^3,\ 2x_0x_1 + 3y_0^2y_1,\ x_1^2 + 2x_2x_2 + 3y_0y_1^2 + 3y_0^2y_2)$$

$$J_n(R) = \mathbb{C}[x_0, y_0, \cdots, x_n, y_n]/I_n$$

where ideal I_n is generated by

$$\sum_{\substack{i,j \geq 0 \\ i+j=m}} x_i x_j - \sum_{\substack{i,j,k \geq 0 \\ i+j+k=m}} y_i y_j y_k$$

What is a... Bifiltered Ring

a bifiltered ring, R, is equipped with a sequence of subgroups

$$F_{p,q}R \subseteq R \quad \forall p, q \in \mathbb{Z}$$

What is a... Bifiltered Ring

a bifiltered ring, R, is equipped with a sequence of subgroups

$$F_{p,q}R \subseteq R \quad \forall p, q \in \mathbb{Z}$$

such that

$F_{p,q}R \subseteq F_{p',q'}R$ if $p \leq p', q \leq q'$

$$\bigcup_{p,q} F_{p,q}R = R$$

$$F_{p,q}R \cdot F_{p',q'}R \subseteq F_{p+p',q+q'}R$$

$\mathrm{HS}_{q,q,T}(\mathrm{gr}(J_\infty(R)))$

What is a... Bifiltered Ring

a bifiltered ring, R, is equipped with a sequence of subgroups

$$F_{p,q}R \subseteq R \quad \forall p, q \in \mathbb{Z}$$

such that

$$F_{p,q}R \subseteq F_{p',q'}R \quad \text{if } p \leq p', q \leq q'$$

$$\bigcup_{p,q} F_{p,q}R = R$$

$$F_{p,q}R \cdot F_{p',q'}R \subseteq F_{p+p',q+q'}R$$

example: $R = \mathbb{C}[x, y]$

$$F_{p,q}R = \{ \text{ polynomials } P(x, y) \text{ with } \deg_{x,y}(P) \leq (p, q) \ \}$$