A Missing Link

Brane Networks and the Cobordism Conjecture

Based on work with:

Ignacio Ruiz

— to appear —

Strings & Geometry

Uppsala— May 19, 2026

Markus Dierigl

Funded by the European Union

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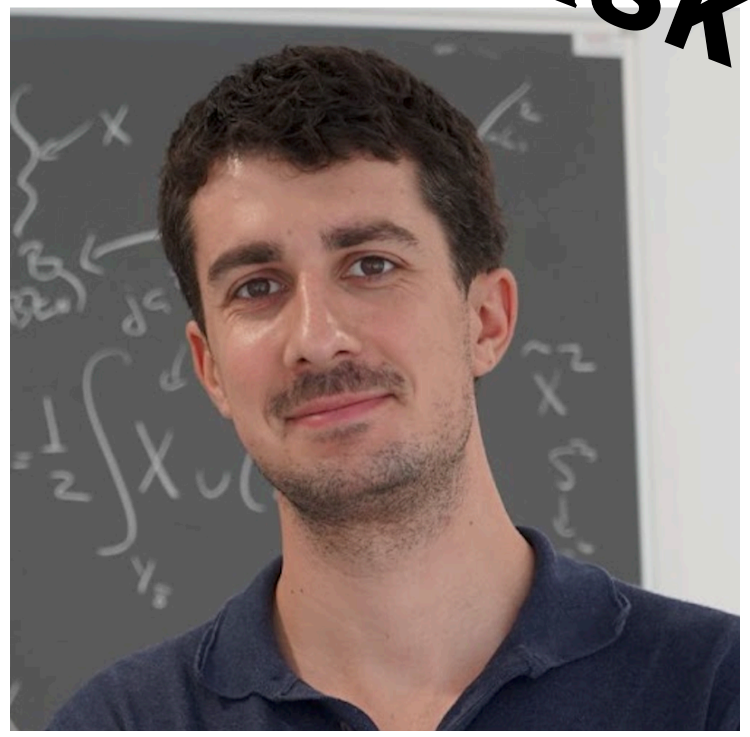
Also ask Nacho!



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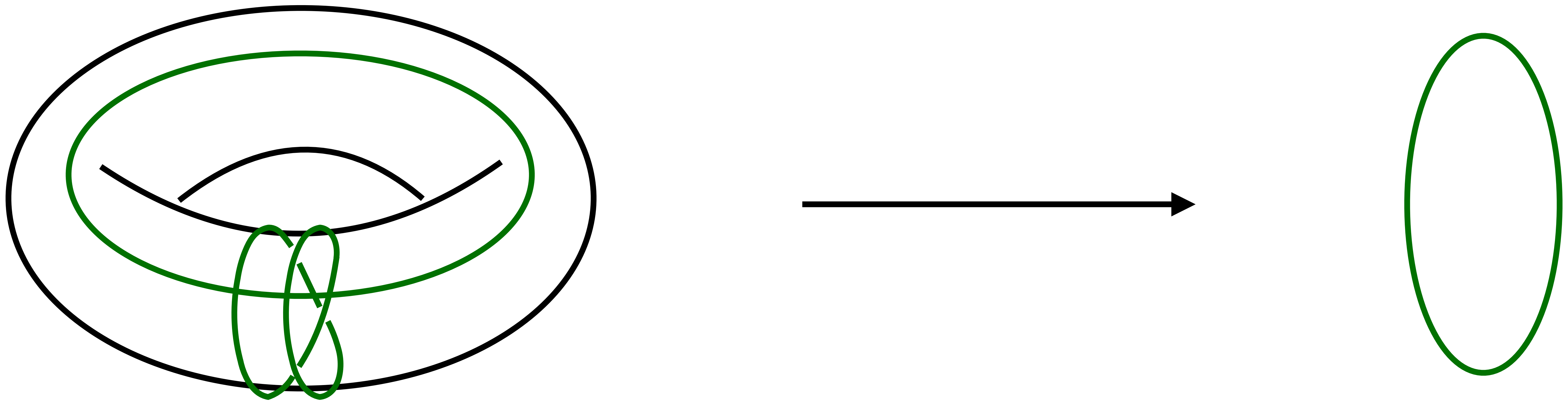
Gauge theories

Gauge bundles are classified by **maps into classifying space** BG

$$X \longrightarrow BG$$



Separates theory into **sectors,** described by homotopy classes

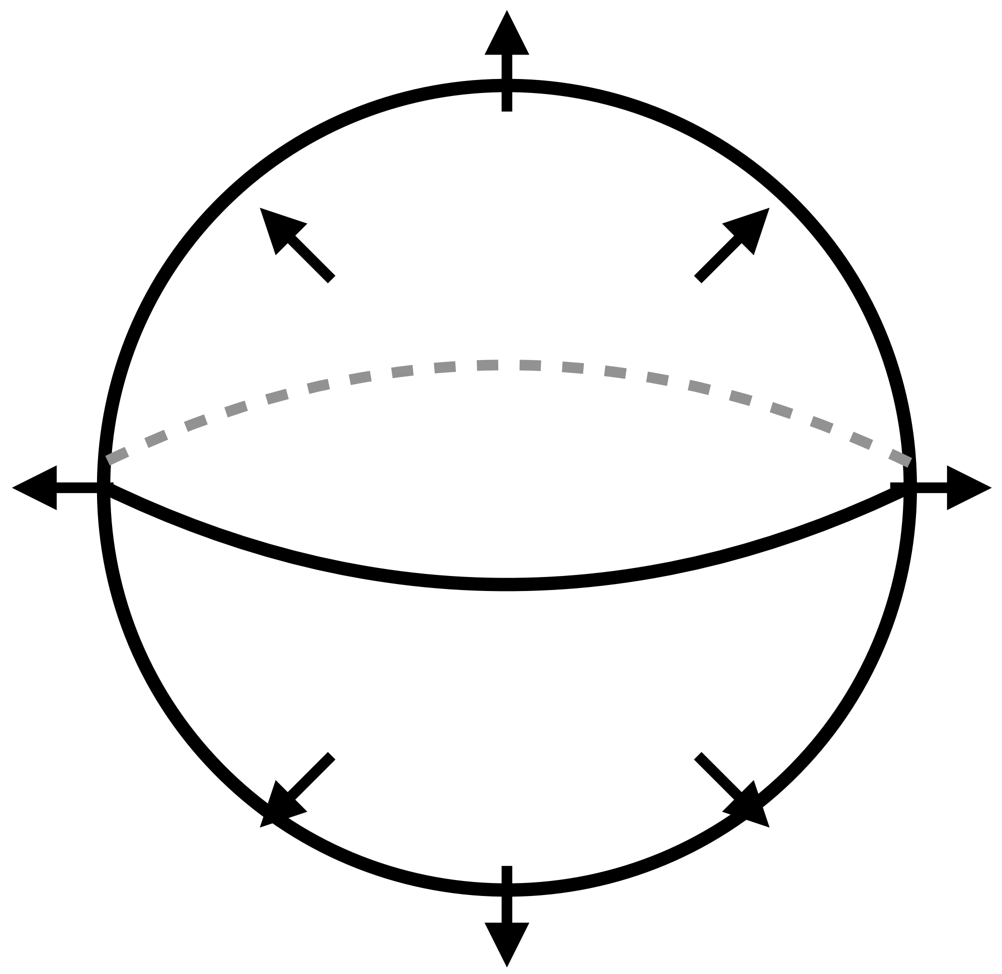
$$[X, BG]$$

Global charges in gauge theories

Gauge theories often come with a class of **global symmetries**

$$\mathcal{L} = \frac{1}{2e^2} F \wedge *F$$

Some are **encoded in topological features of bundle** $X \to B\mathrm{U}(1)$



$$d(F \wedge \cdots \wedge F) = 0$$

conserved currents

magnetic symmetries

similar for higher-form fields

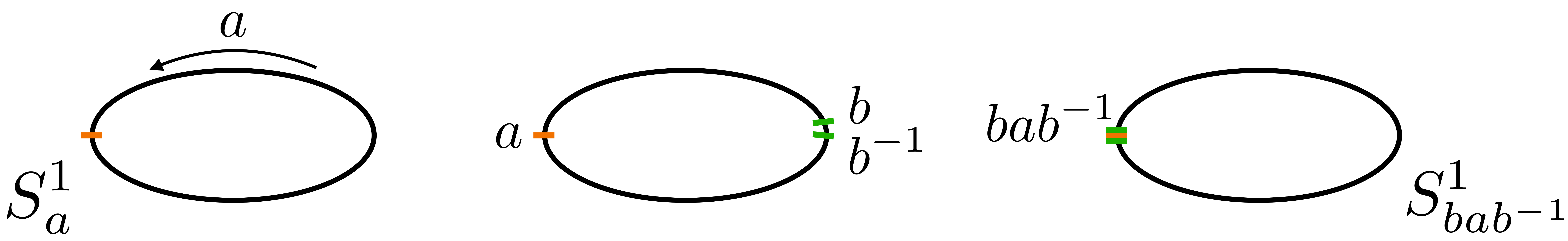
See, e.g., [Heidenreich, McNamara, Montero, Reece, Rudelius, Valenzuela'19], [Garcia-Valdecasas, Reece, Suzuki '24], [MD, Minasian, Novicic '26]

Today: Discrete gauge symmetry

Topological classes of G bundle specified by

$$[X, BG] \quad \rightarrow \quad \frac{\mathrm{Hom}\big(\pi_1(X); G\big)}{G}$$

In particular on circle

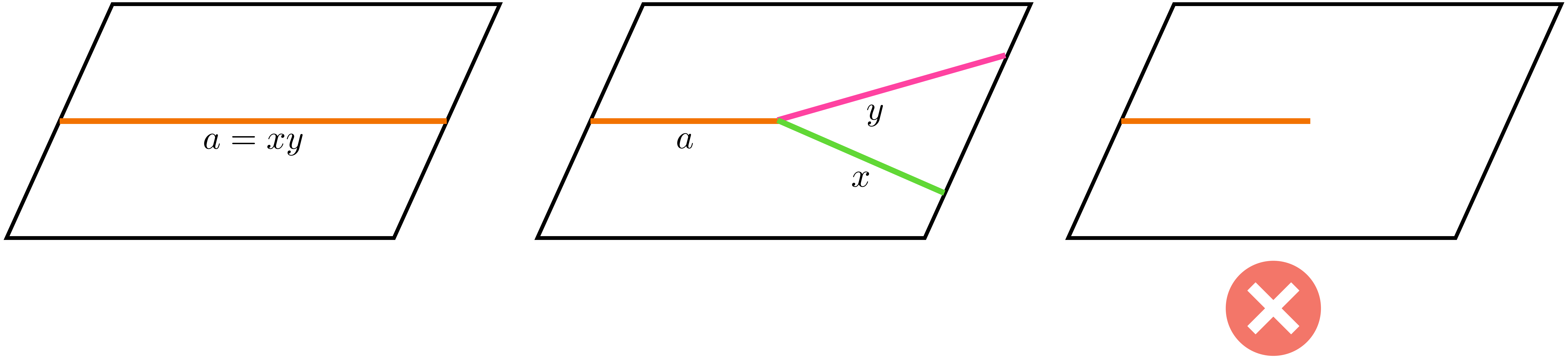


Has to **respect group properties of** $\pi_1(X)$

Transition functions

The (dual of the) **transition functions are the topological operators**

$$a\,, x\,, y \in G$$

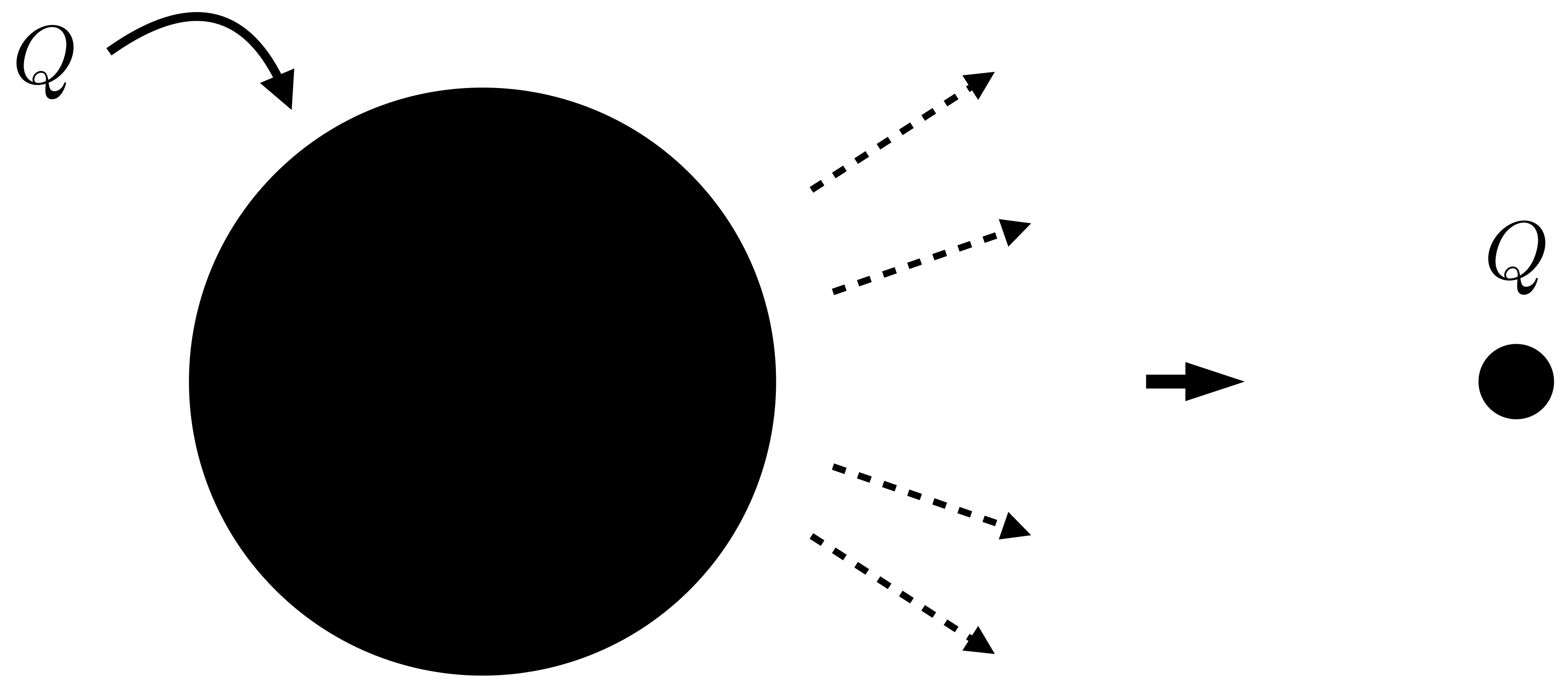


group-like fusion (interesting non-invertible if it goes beyond)

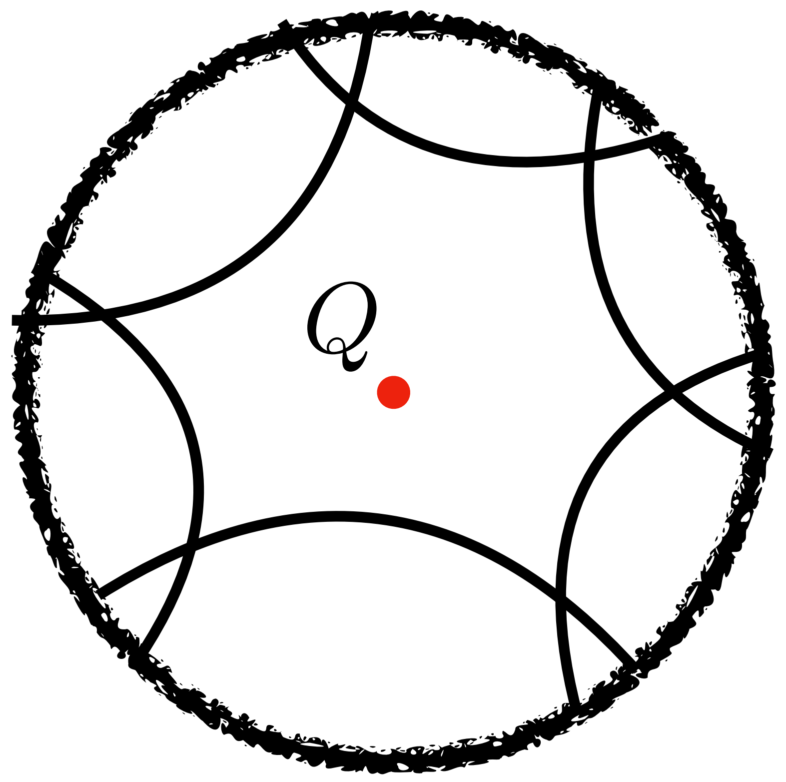
→ **but cannot end**

Switching on (quantum) gravity

See e.g. [Banks, Dixon ’88], [Banks, Seiberg ’11], [Harlow, Ooguri ’18], [Harlow, Shaghoulian ’20], [Bah, Chen, Maldacena ’22], [Heckman, Hübner, Murdia ’24], …



AdS/CFT



violation of **entropy bounds**

↓

No global symmetries

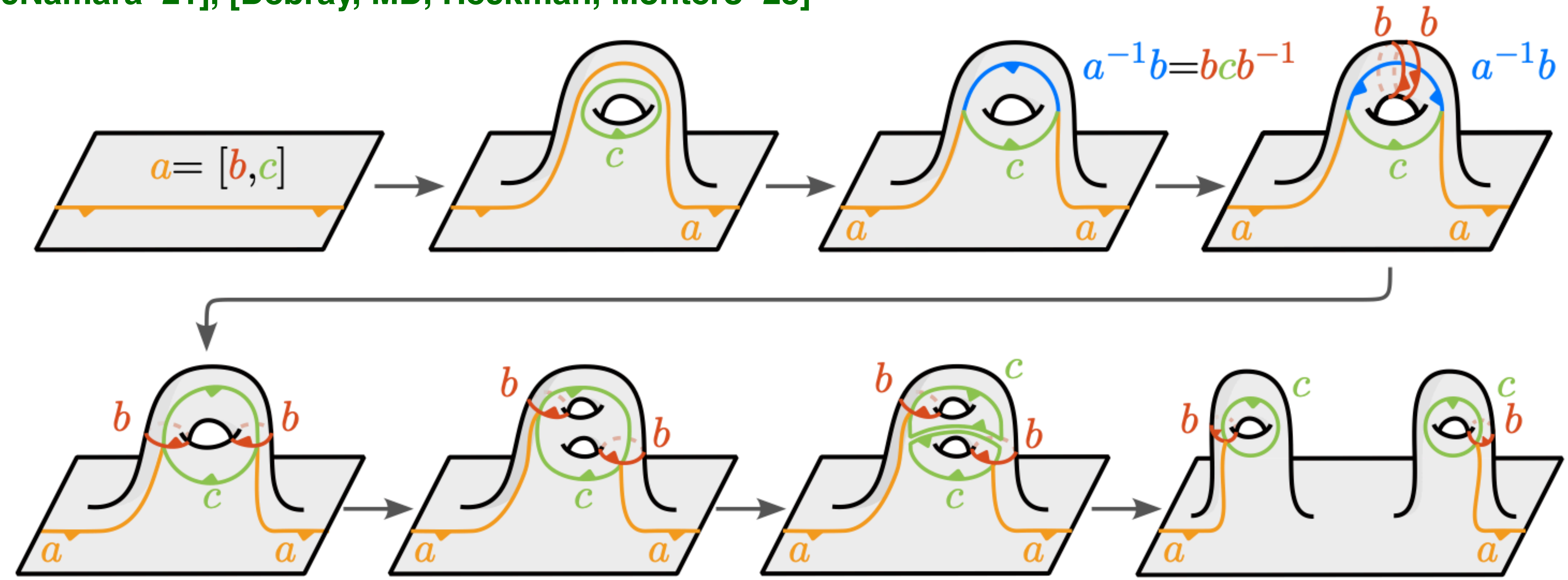
↑

Violation of bulk-boundary correspondence

Gravitational solitons

They can ‘split’ transition functions

[McNamara ’21], [Debray, MD, Heckman, Montero ’23]



➙ **Price to pay: topological complexity of spacetime**

see also [Ruiz ’24], [Las Heras, Ruiz ’26]

Cobordism Conjecture

[McNamara, Vafa '19]

What is left?

$$\boxed{\Omega_k^{\xi}(BG)}$$

Deformation classes that **cannot be 'resolved'** in smooth background (described by **bordism groups**)

→ **Conserved charges** (global symmetries)

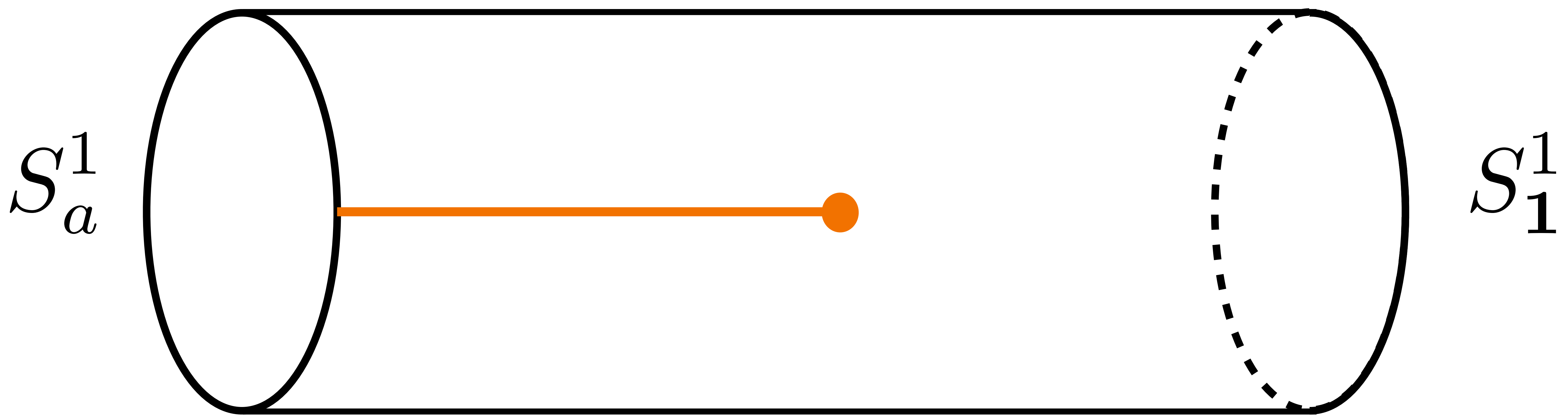
In Quantum Gravity:

$$\boxed{\Omega_k^{\mathrm{QG}} = 0}$$

Symmetry-breaking defects

Sometimes this is **not true in an approximation to Quantum Gravity**

➔ requires a **symmetry-breaking defect** (singular in approximation)

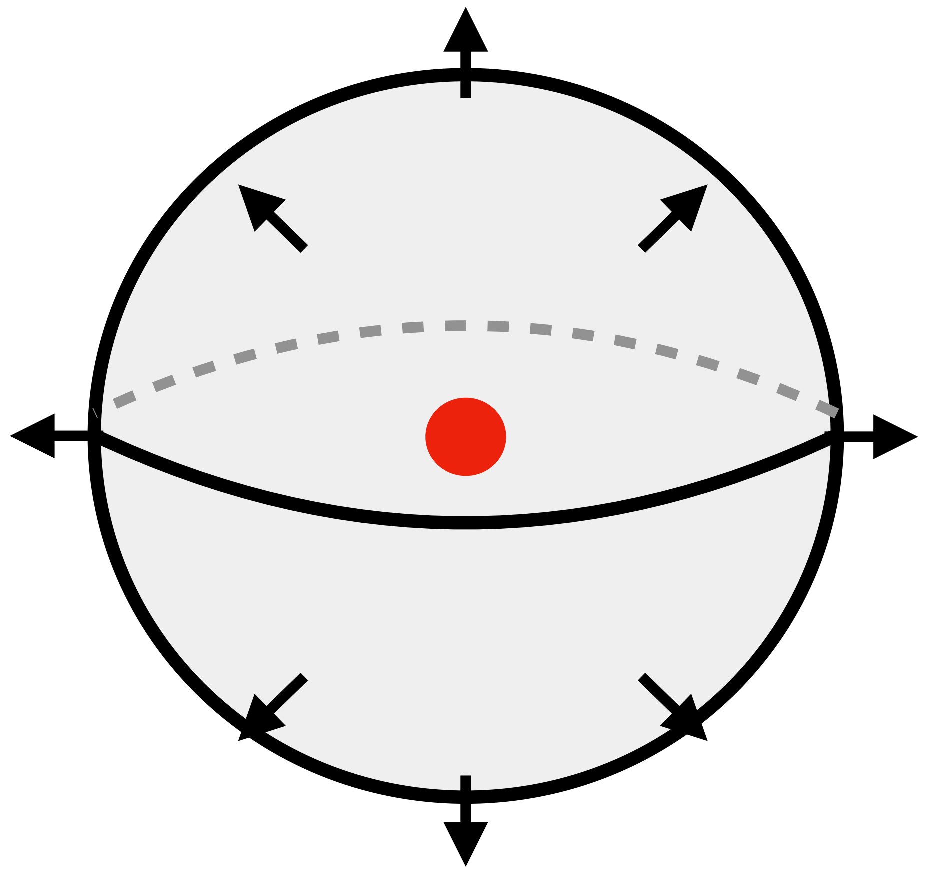


it's asymptotics are described by **generators of** $\Omega^{\xi}_{k}(BG)$

(gauging typically introduces new symmetries) **[MD, Minasian, Novicic '26]**

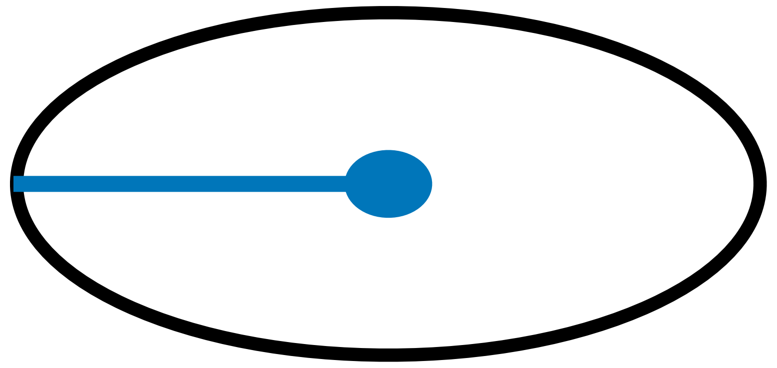
Dimensionality of the object

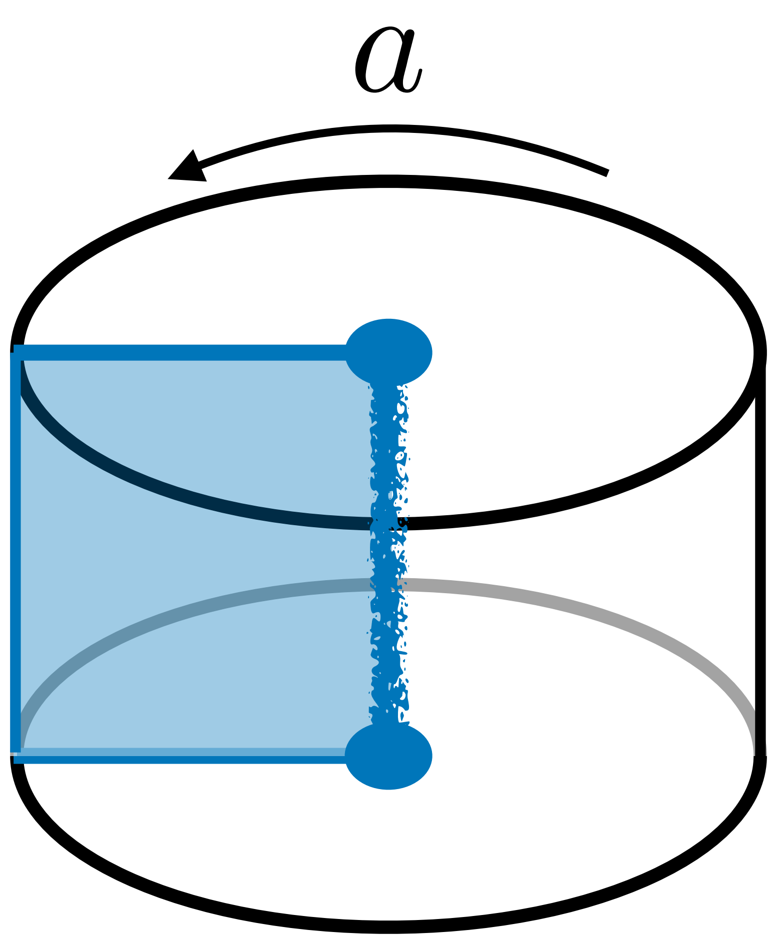
If the **generator is in** $\Omega_k^{\xi}(BG)$ **defect in codimension** $(k+1)$



$$\Omega_{p+1}^{\mathrm{Spin}}(B^p\mathrm{U}(1)) \supset \mathbb{Z}$$

(e.g., monopole: codim-3)





$$\widetilde{\Omega}_1^{\xi}(BG) = \mathrm{Ab}(G)$$

(codimension-two defect)

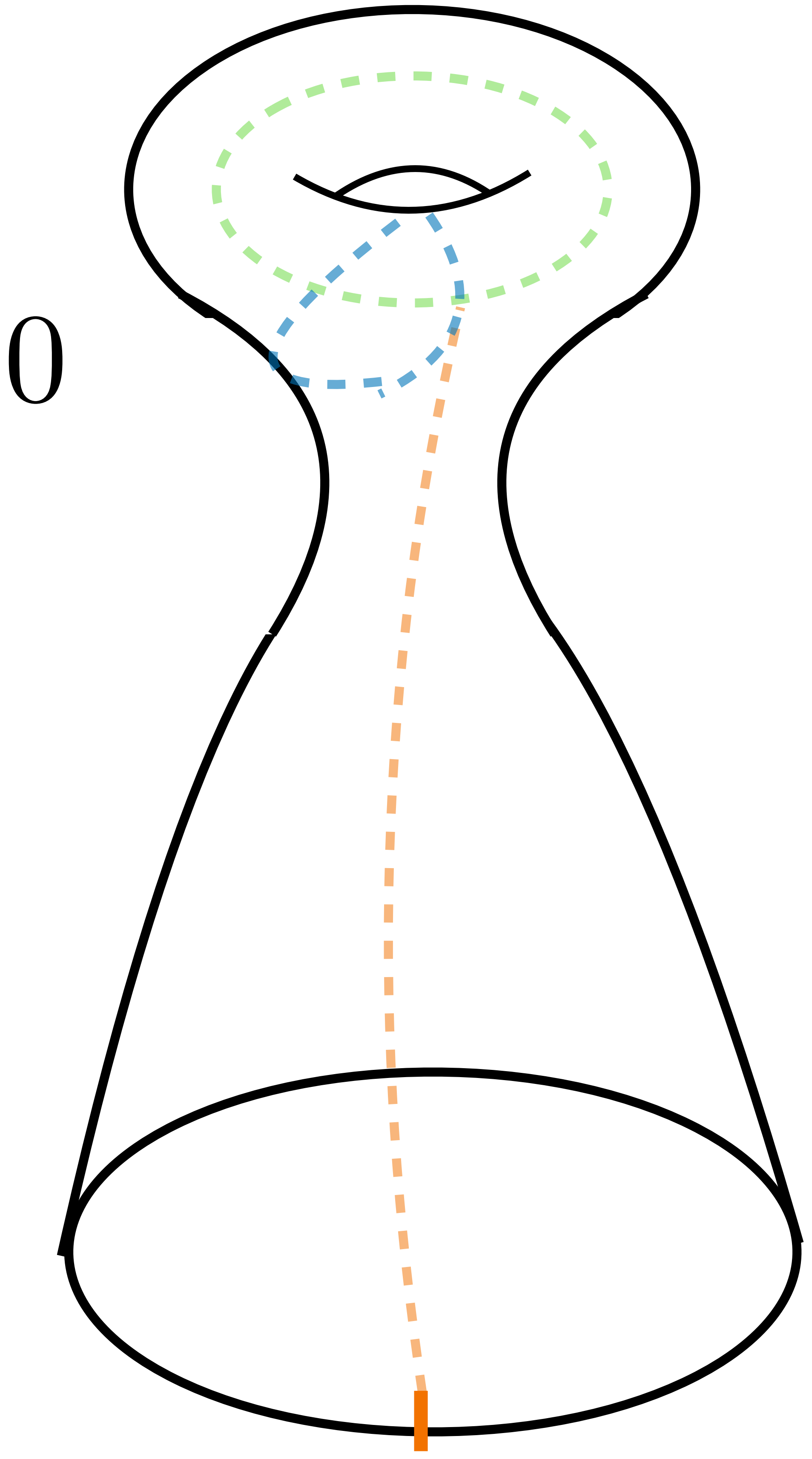
labeled by $a \in G$

A mismatch

[Braeger, Debray, MD, Heckman, Montero ’25]

Example: $$G = \mathrm{SL}(3;\mathbb{Z})$$

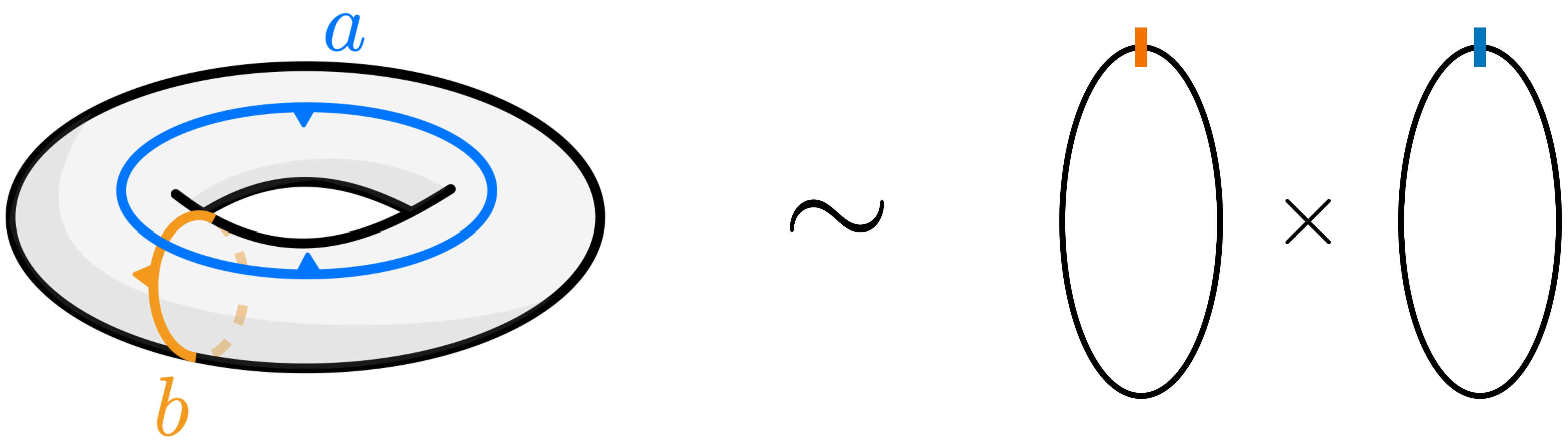
$$\Omega_1^{\xi}\big(B\mathrm{SL}(3;\mathbb{Z})\big) \supset H_1\big(B\mathrm{SL}(3;\mathbb{Z});\mathbb{Z}\big) = \mathrm{Ab}\big(\mathrm{SL}(3;\mathbb{Z})\big) = 0$$



any $a \in \mathrm{SL}(3;\mathbb{Z})$

But one has:

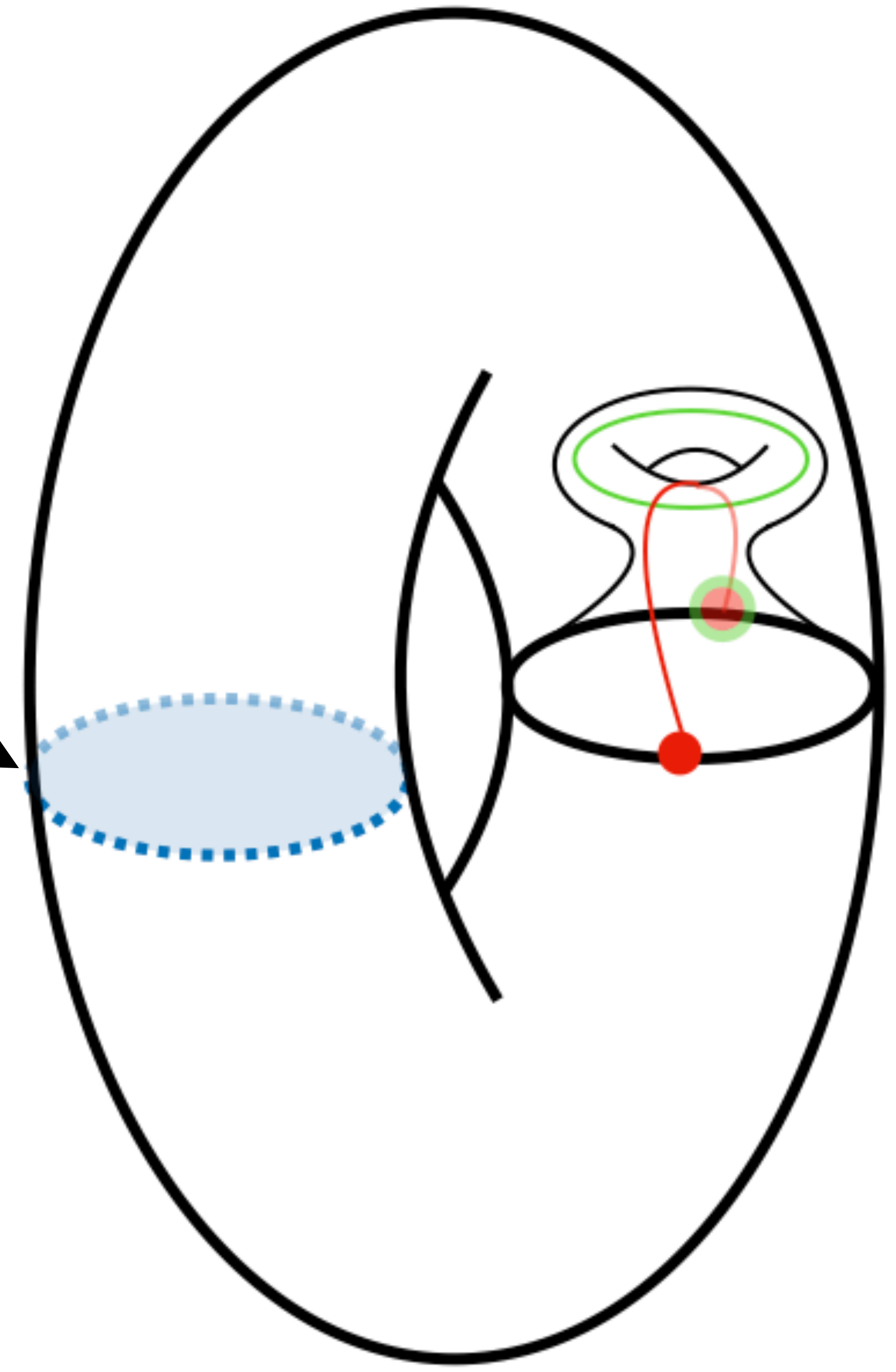
$$\Omega_2^{\xi}\big(B\mathrm{SL}(3;\mathbb{Z})\big) \supset H_2\big(B\mathrm{SL}(3;\mathbb{Z});\mathbb{Z}\big) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$$



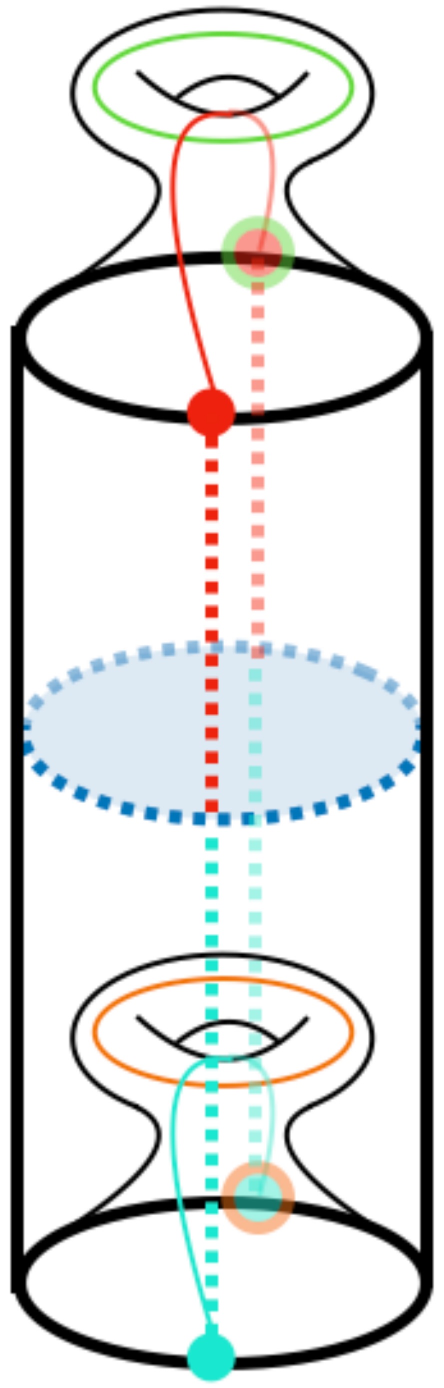
Why don’t the solitons work?

Try to use the soliton

b transition function



$a = [x, y]$



action incompatible with

for any choice of x , y

→ **does not lead to well-defined background**

Hopf's theorem

Are all generators of this type?

Given a **group presentation of G:**

$$G = \frac{F}{R}$$

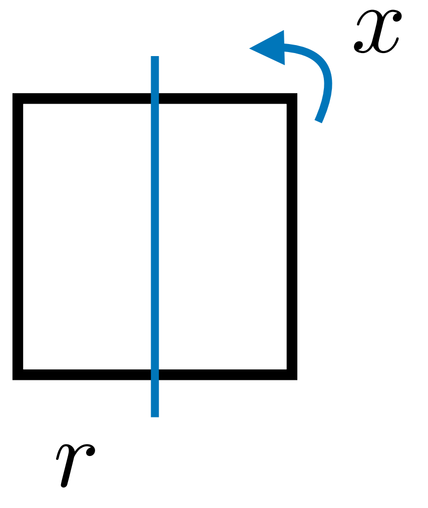
free group

(normal closure of) **relations**

Examples:

$\langle x | x^n \rangle$ cyclic group $\mathbb{Z}_n$

$\langle x, r | \{x^n , r^2 , rxrx\} \rangle$ dihedral group D_{2n}



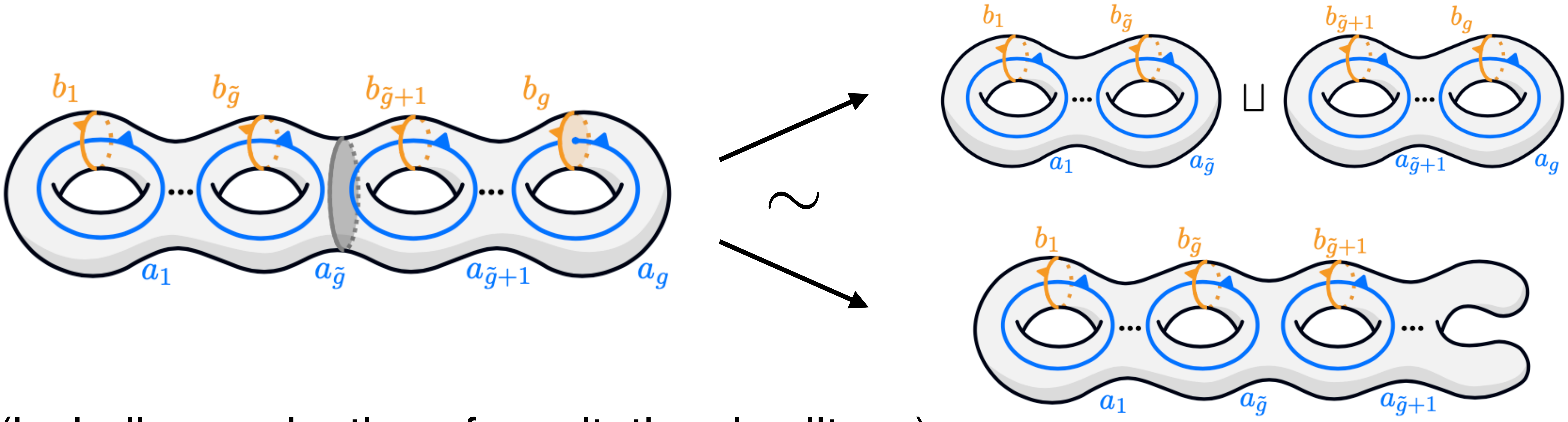
$$\boxed{\rightarrow \quad H_2(BG;\mathbb{Z}) = \frac{R \cap [F,F]}{[R,F]}}$$

Guaranteed to form a subset of $\Omega_2^{\xi}(BG)$ $(\xi = \mathrm{SO}\,, \mathrm{Spin}\,)$

The generators

Each element in $H_2(BG;\mathbb{Z})$ **associated to**

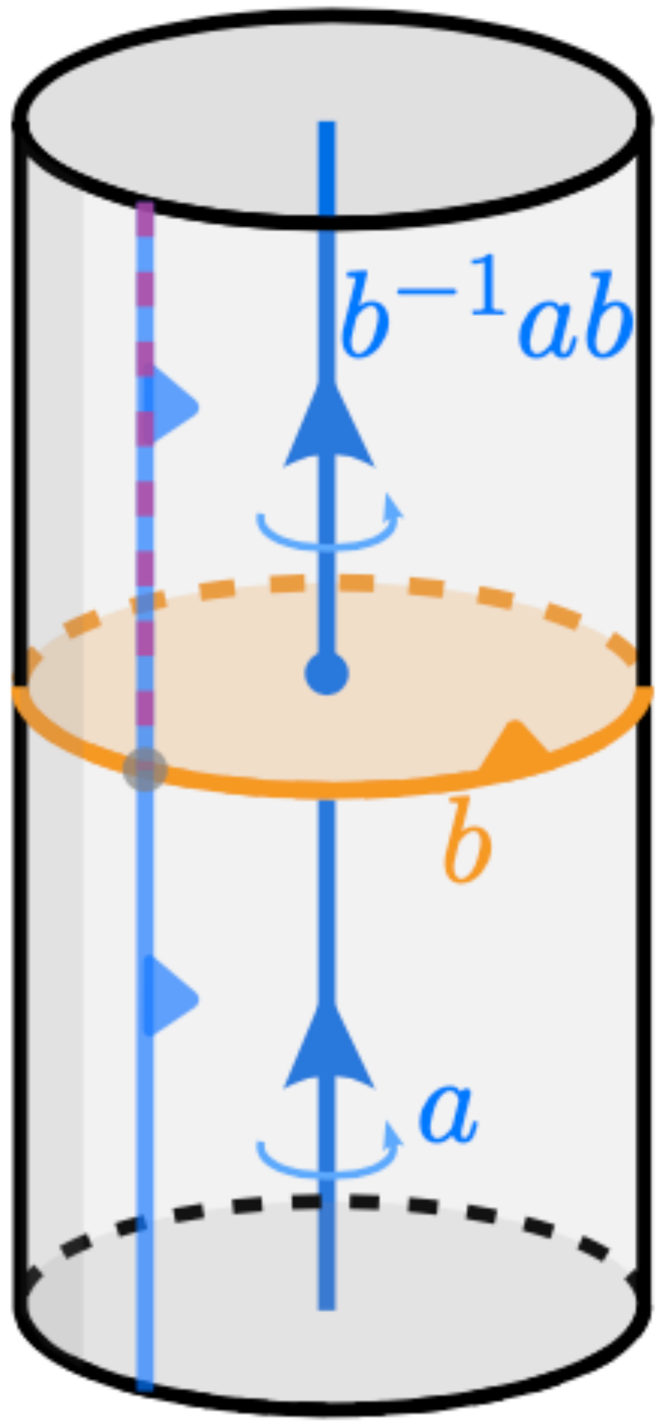
$$\boxed{[a_1,b_1]\dots[a_g,b_g]=1} \qquad R\cap[F,F]$$



(including nucleation of gravitational solitons)

Obstructions revisited

Not an **isolated brane**, but:



codimension-two defect passing through a transition function (codimension three)

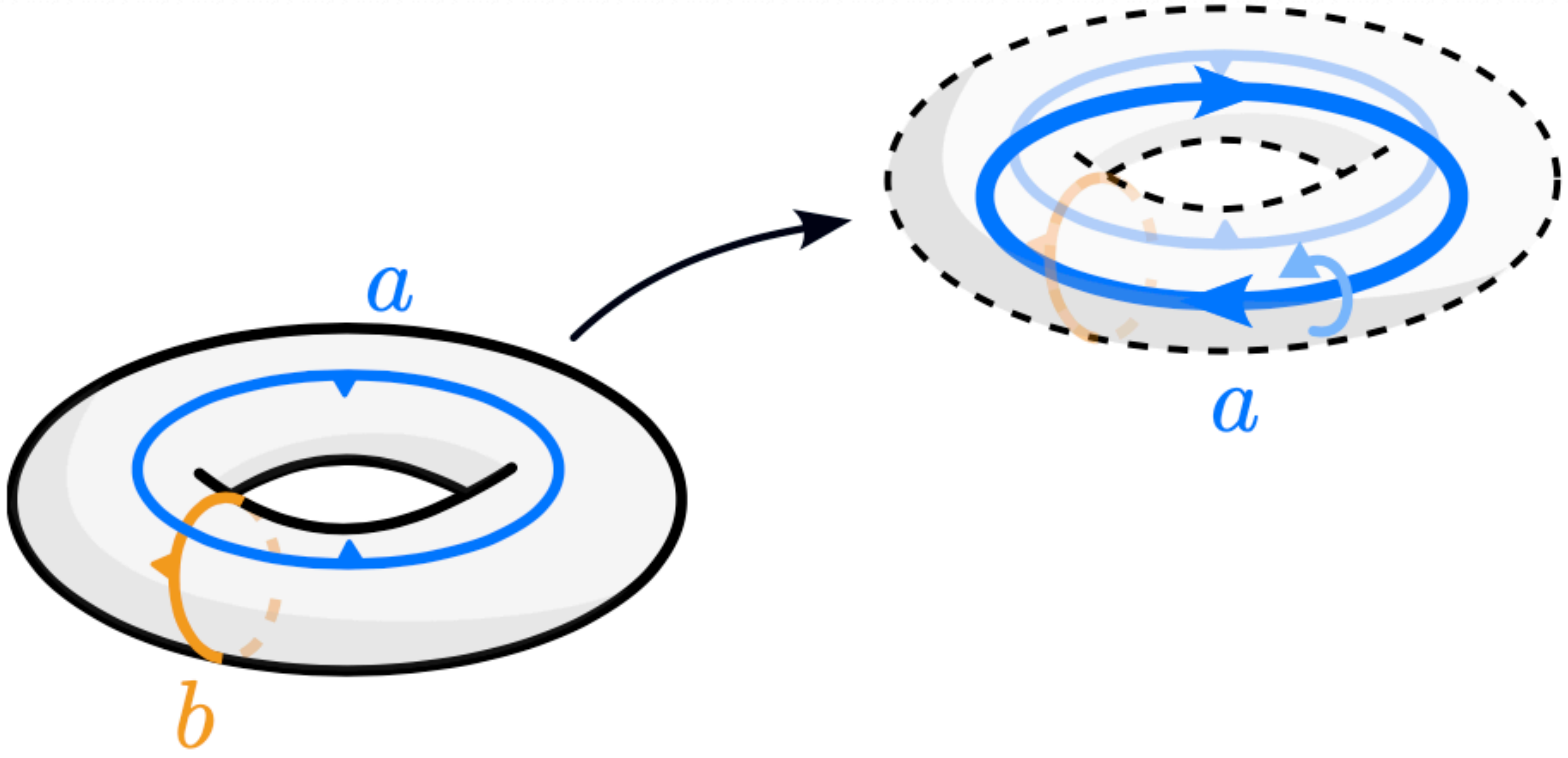
➛ **naturally in codimension three**

More **intuitive picture** in terms of defects?

A link

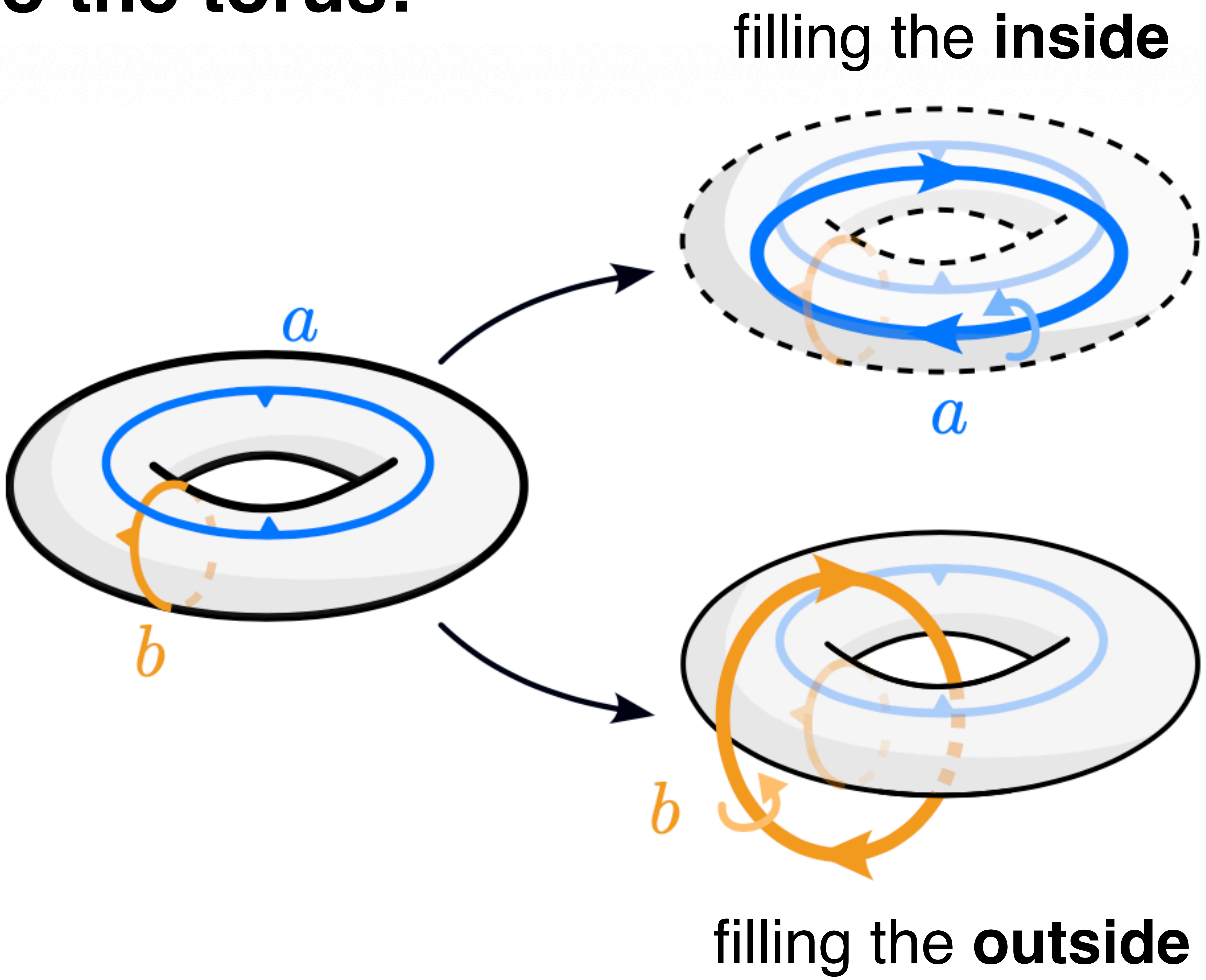
Back to the torus:

filling the **inside**



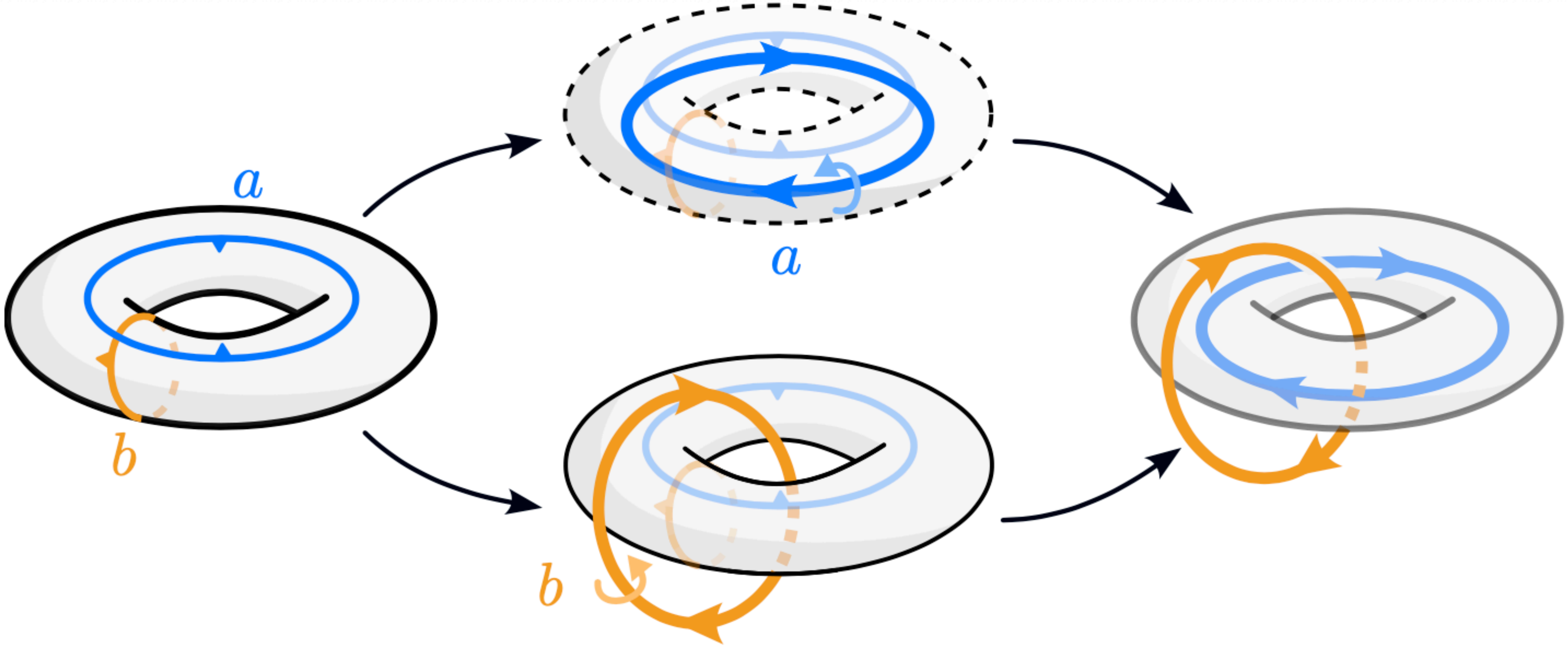
A link

Back to the torus:



A link

Back to the torus:

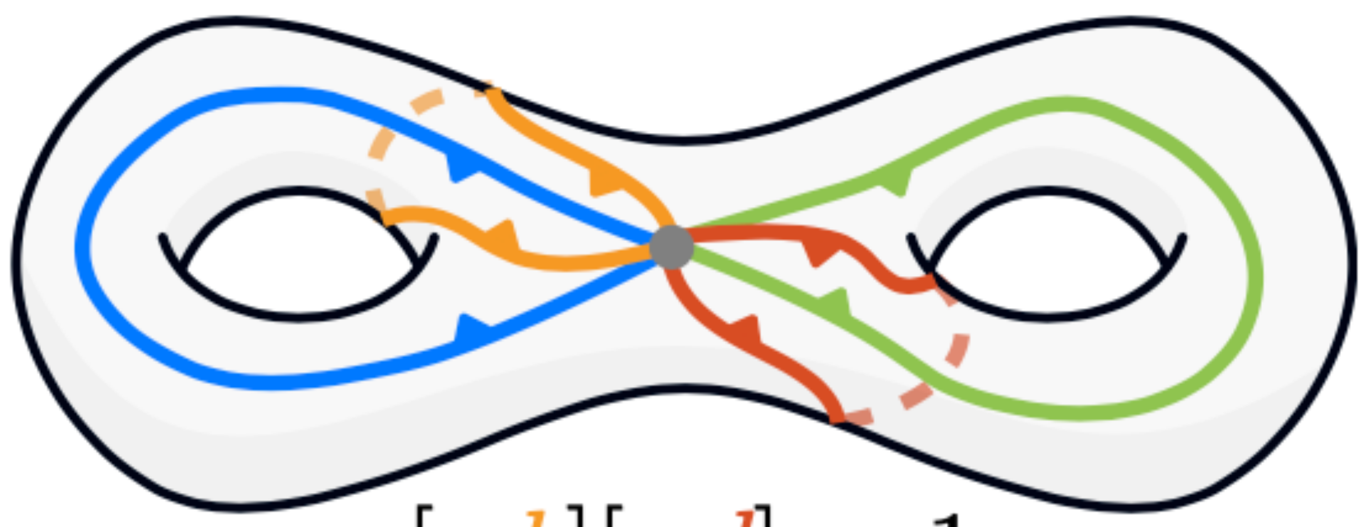


➔ Linking of codimension-two defects

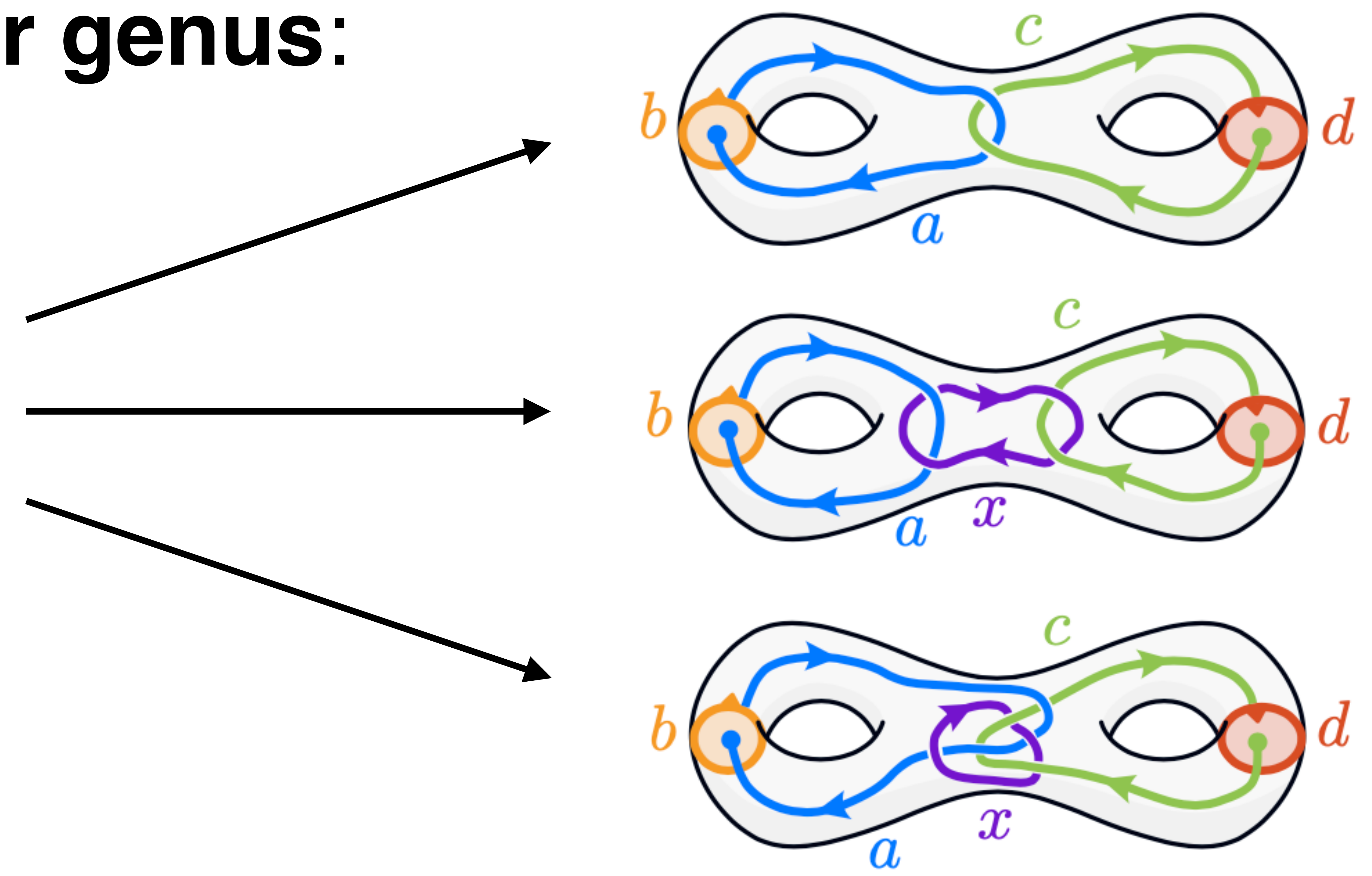
A link inside?

Is the defect always the link of codimension-two defects?

Let's have a look at **higher genus**:



$[a,b][c,d] = 1$



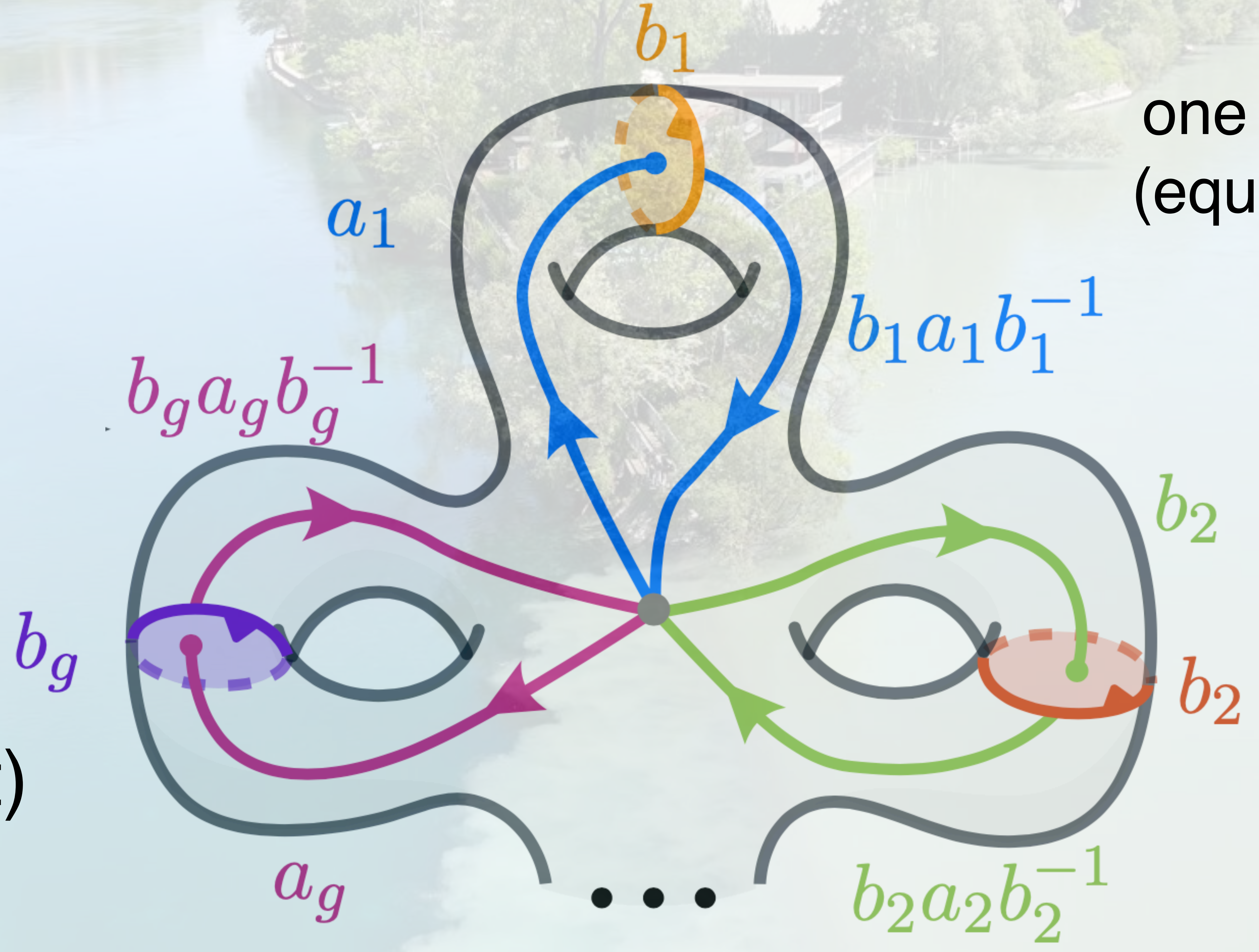
potential links

But for **each component** has to satisfy **condition** (for defects to close)

➞ **must be part of relations** R **otherwise no solution**

A junction

What always works: 2g-valent junction



one can also fill **outside** (equivalent defect, linked with one inside)

junctions might be simplified (model-dependent)

→ **Prediction of junctions** (properties of defects)

Example: Type IIA on a Calabi-Yau

Effective theory is $\mathcal{N} = 2$ **supergravity and contains**

$$\int B_2 \wedge dC_3 \wedge dC_3 \xrightarrow[2(h^{2,1}+1)]{\text{on 3-cycles}} \int B_2 \wedge d\chi_I \wedge d\tilde{\chi}^I$$

dualize to σ

Discrete **axionic shift symmetries mix:**

$$\begin{cases} \chi_I \to \chi_I + \zeta_I \,, \\ \tilde{\chi}^I \to \tilde{\chi}^I + \tilde{\zeta}^I \,, \\ \sigma \to \sigma + \alpha - \frac{1}{2}(\zeta_I \tilde{\chi}^I - \tilde{\zeta}^I \chi_I) \,, \end{cases}$$

Example: Type IIA on a Calabi-Yau

Induced **discrete symmetry: Heisenberg group**

$$\mathrm{H}_{2h^{2,1}+3}(\mathbb{Z}) = \langle a_0, b_0, \ldots, a_{h^{2,1}}, b_{h^{2,1}}, z \mid [z, a_I], [z, b_I], [a_I, a_J], [b_I, b_J], [a_I, b_J] z^{-\delta_{IJ}} \rangle$$

$$\sigma \qquad\qquad \chi_I \qquad \tilde{\chi}^I$$

Rich **homology groups:**

$$H_2\big(B\mathrm{H}_{2h^{2,1}+3}(\mathbb{Z}); \mathbb{Z}\big) = \begin{cases} \mathbb{Z}^2 & \text{for } h^{2,1} = 0\,, \\ \mathbb{Z}^{2h^{2,1}(h^{2,1}+1)} \oplus \mathbb{Z}^{h^{2,1}} & \text{for } h^{2,1} \geq 1\,. \end{cases}$$

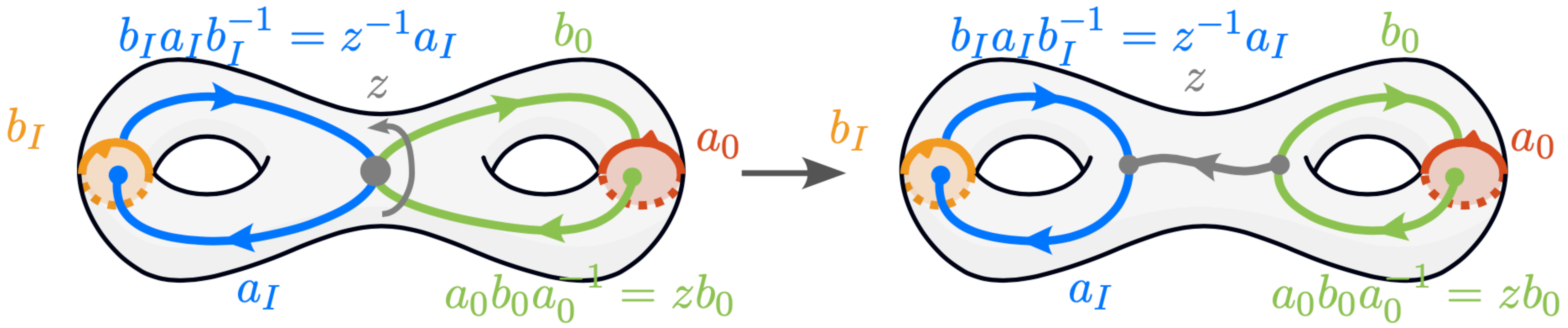
(Caveat: potential model-dependent mixing with other groups)

Example: Type IIA on a Calabi-Yau

Especially interesting: $[a_I, b_I]z^{-1} = 1$

$\rightarrow$ $[a_I, b_I][a_J, b_J] = 1$ **genus two**

Strings from wrapped D4-branes (on 3-cycles) **and fundamental**



Here: resolution to two **trivalent junctions** possible

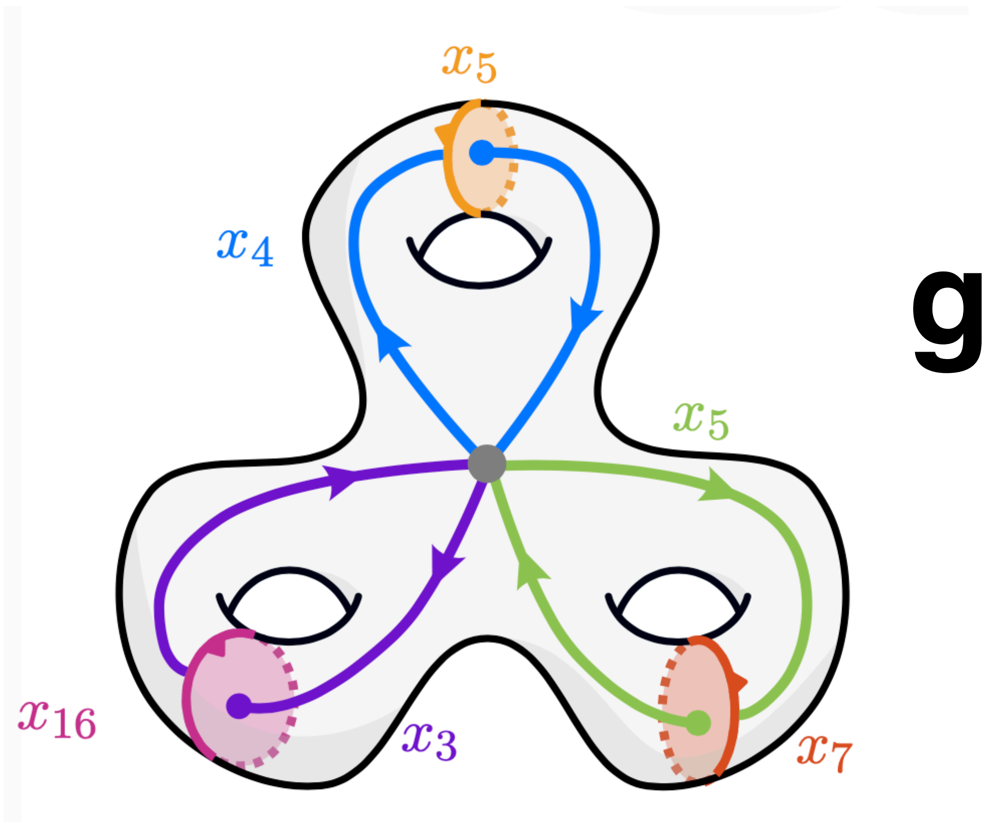
Generalizations

Consider **M-theory on 7-manifold**, in presence of topological term

$$S_{\mathrm{M}} \supset -\frac{1}{6}\int C_3 \wedge G_4 \wedge G_4$$

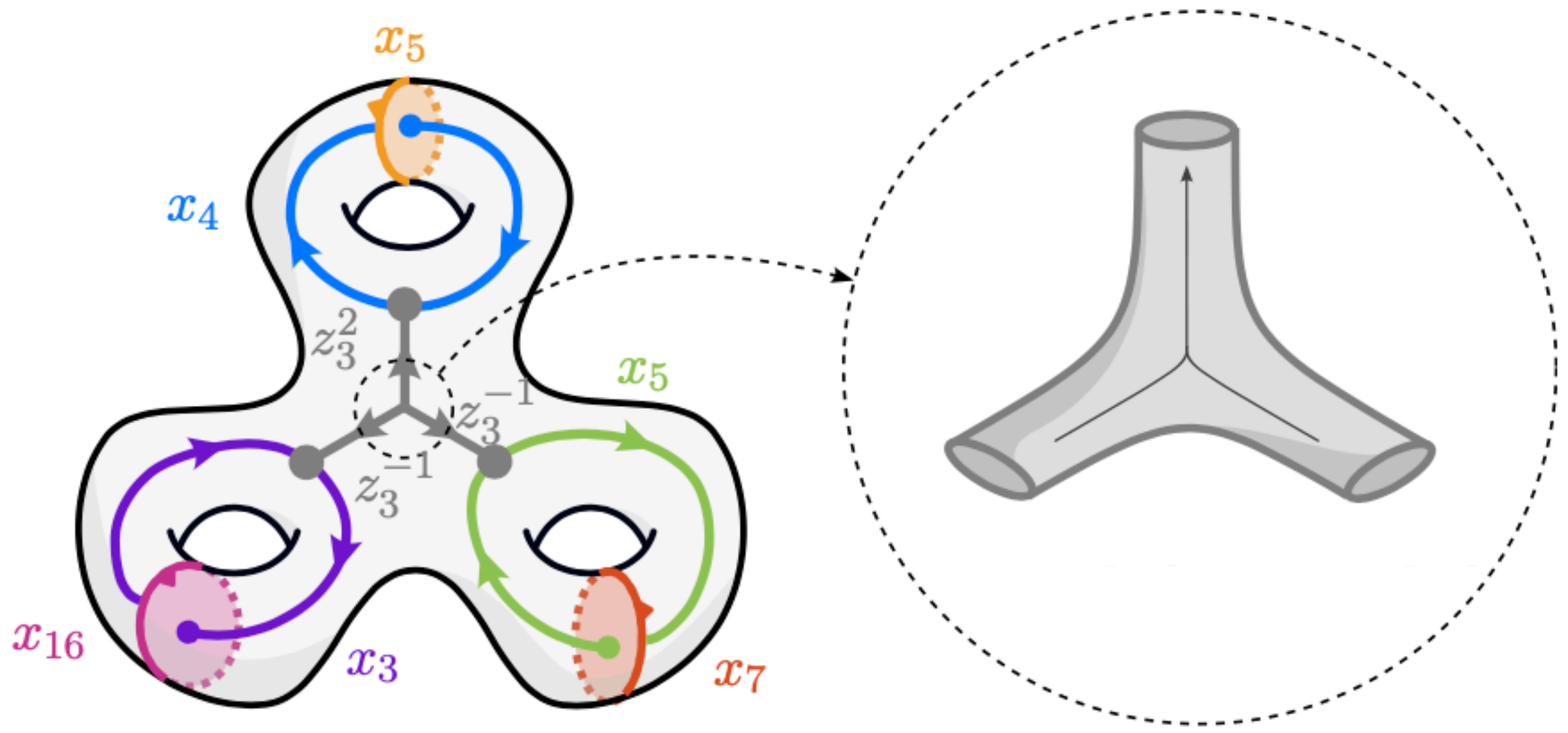
mixing of 4d axions (more/torsional z`s) controlled by intersections

Proof of principle: twisted 7-torus (supersymmetry broken completely)



genus-3

can be resolved



Conclusions

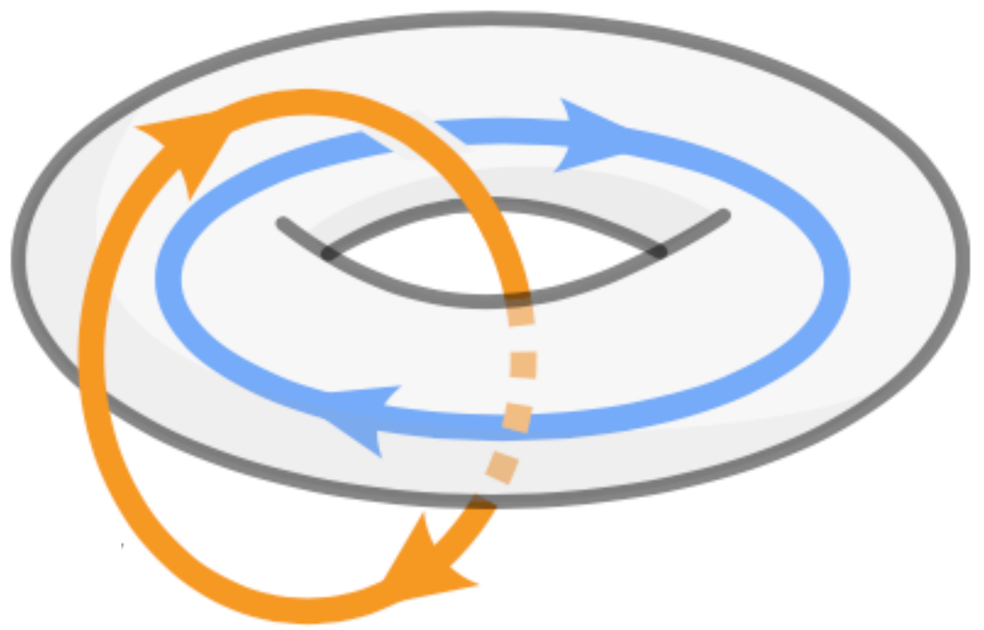
Cobordism Conjecture knows about **brane configurations:**

explains dimension count

$$\Omega_2^\xi(BG)$$

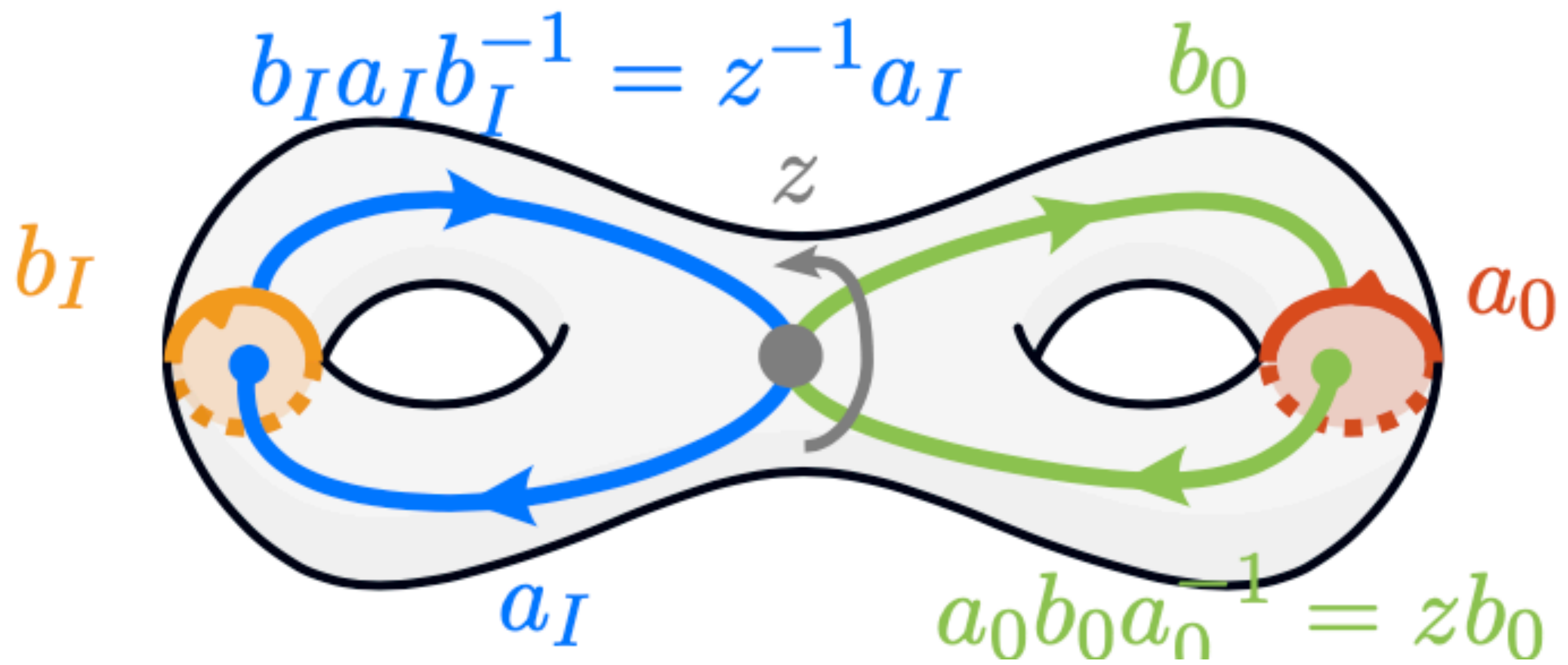
$g = 1$

linking configurations



$g \geq 2$

junctions / linking



Realized **in string theory compactifications** (wrapped branes/strings)

Outlook

- What **groups** are possible and what is their **origin**?
  [Baykara, MD, Kim, Vafa, Xu '25]

- What about **objects in higher codimension**?
  see also [Nevoa, Raman, Vafa '25]

fibration structure

- Encoding **higher-group structures** (mixing with spacetime)?
  [Chakrabhavi, Debray, MD, Heckman '25]

- How is this **implemented** on the level of **spectral sequences?**