Gaillard-Zumino (GZ) non-invertible symmetries

Based on 2510.18997 with Luca Martucci

Fabio Apruzzi (University of Padova) - Strings and Geometry - Uppsala - May 2026

Generalized Symmetries what for?

Generalized symmetries are ubiquitous in physics, from condensed matter to the standard model.

Direct applications:

- Classification of phases of matter (Landau paradigm/dream). Phases are described in terms of generalized symmetries realization (preservation, spontaneous breaking…) and their anomalies
- Novel selection rules on correlation functions

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Hierarchy of symmetries $\Gamma^{(p_i)}$ (nested structure)

$$\Gamma^{(p_1)} \Leftarrow \Gamma^{(p_2)} \ldots$$

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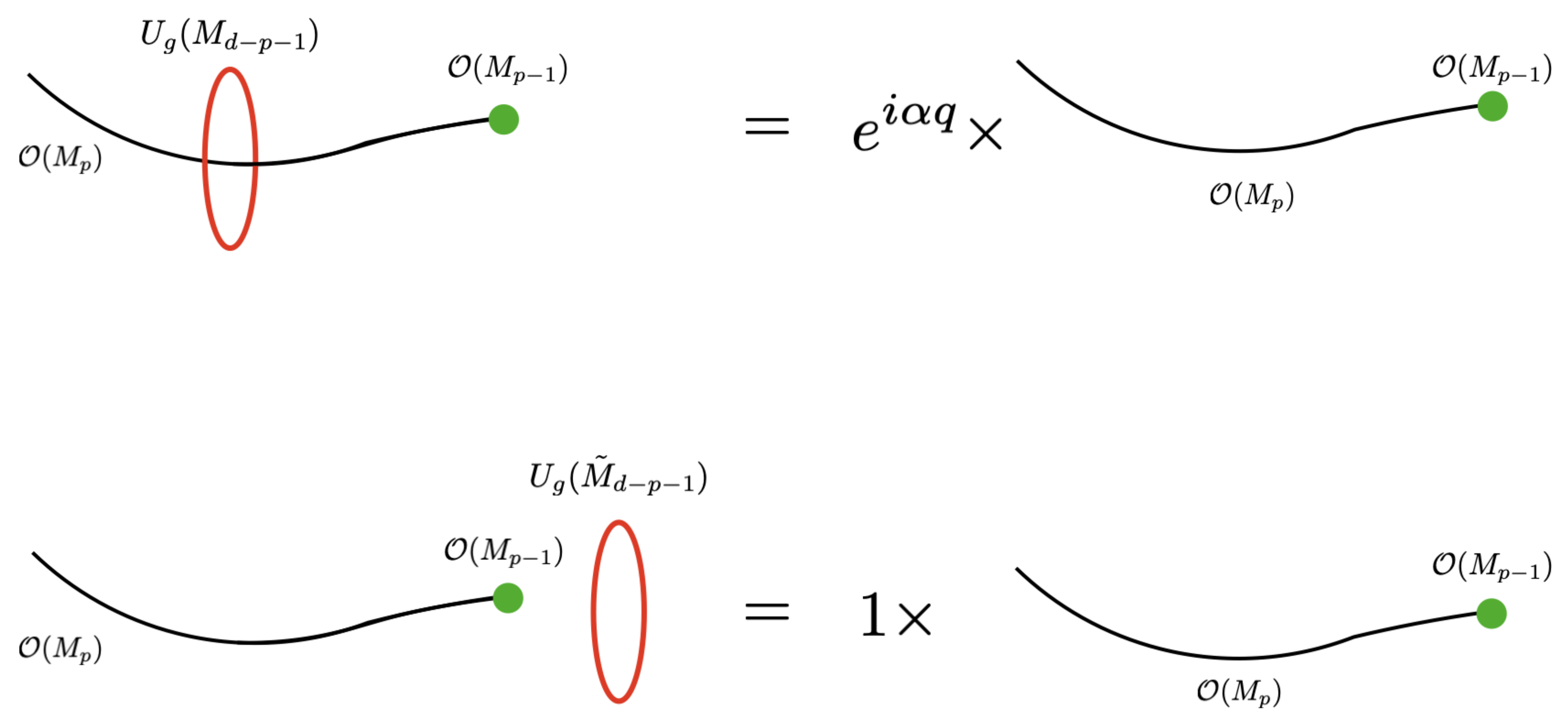
This nested symmetry structure has phenomenological applications:

- Symmetries do not exist in quantum gravity, they can emerge at low energy scales in the effective field theory, they are broken by higher order operators or by screening via light charged objects



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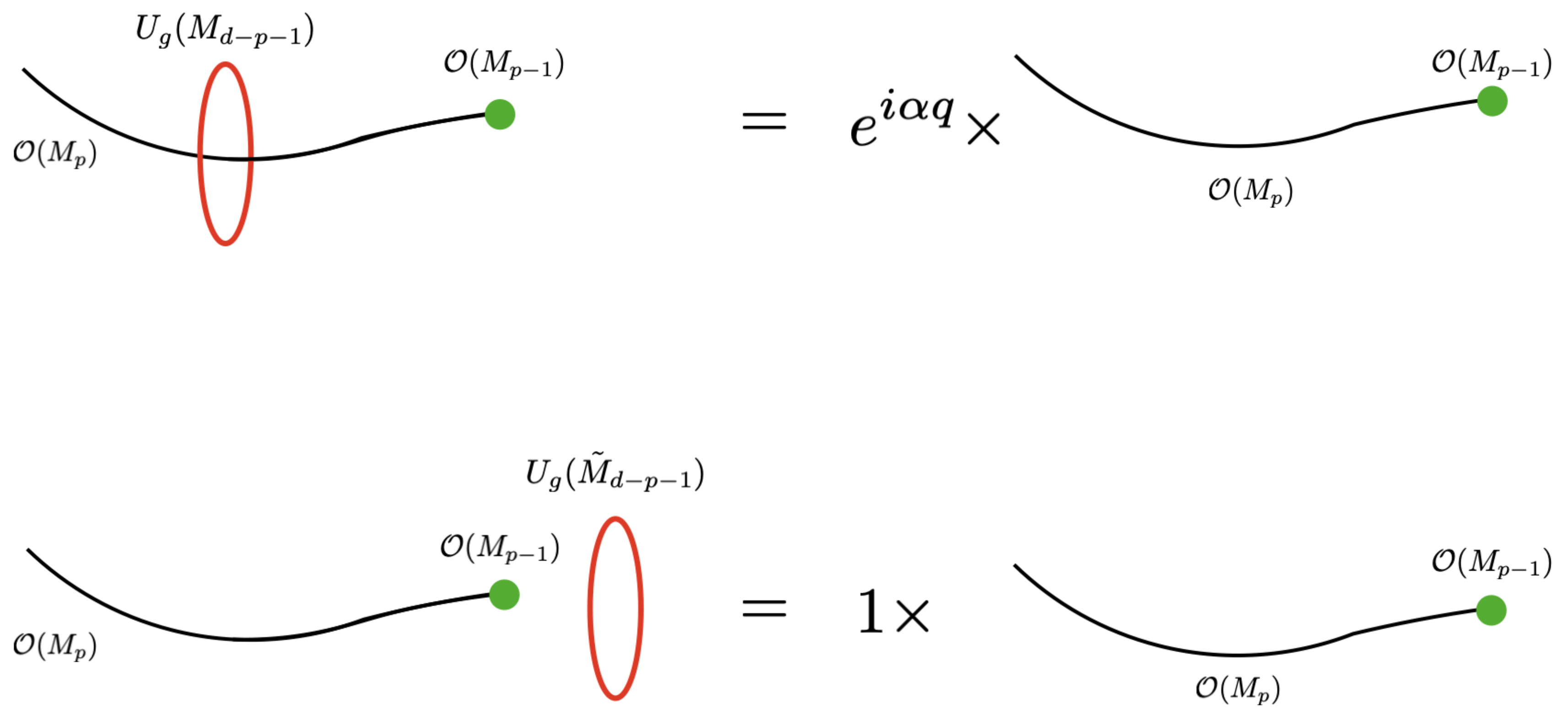

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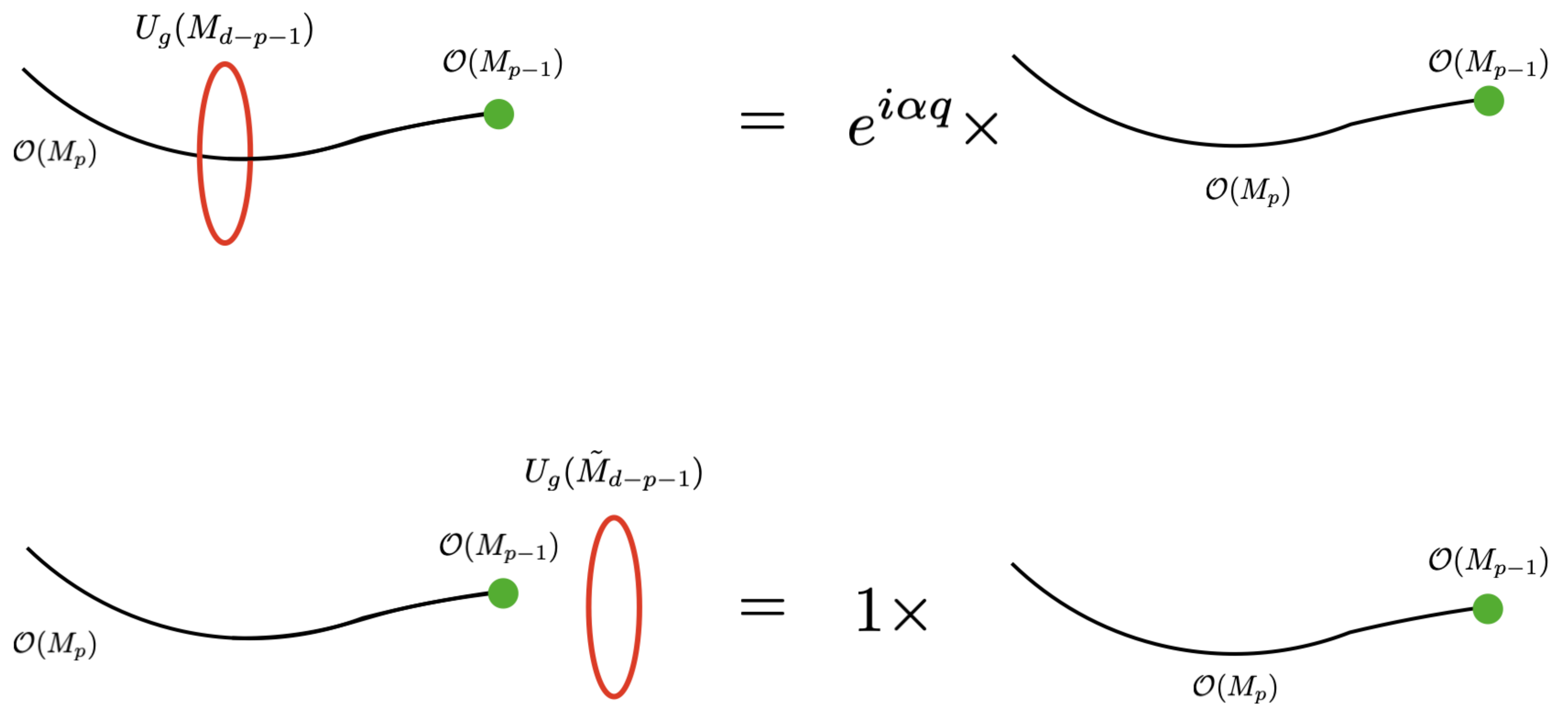

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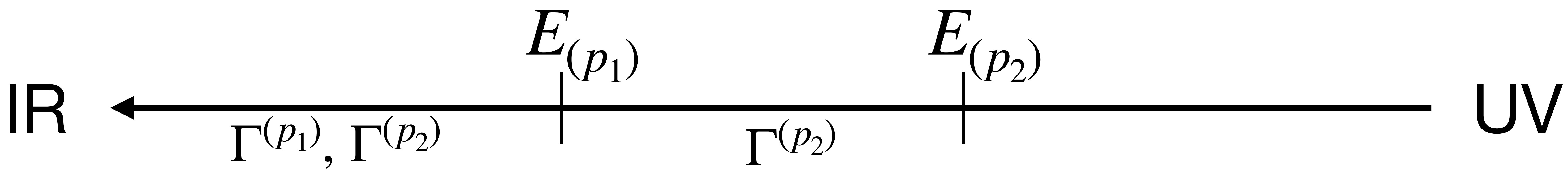


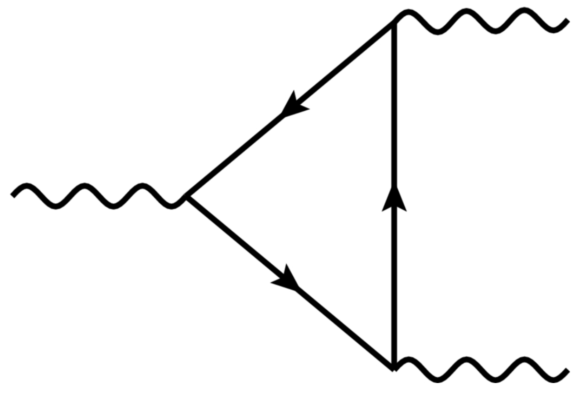
[Heidenreich, McNamara, Montero, Reece, Rudelius, Valenzuela 20,21; Rudelius, Shao 20; Cordova, Ohmori, Rudelius 22; Brennan, Cordova 20 …]

Non-invertible sym in 4d QED

[Choi, Lam, Shao 22; Cordova, Ohmori 22]

ABJ anomaly, tells us that a $U(1)_A$ symmetry is broken quantum mechanically, due to

$$d \star j_A = \frac{1}{8\pi^2} F \wedge F = \frac{1}{2} \star J_m^{(1)} \wedge \star J_m^{(1)} \qquad F = dA = 2\pi \star J_m^{(1)},$$



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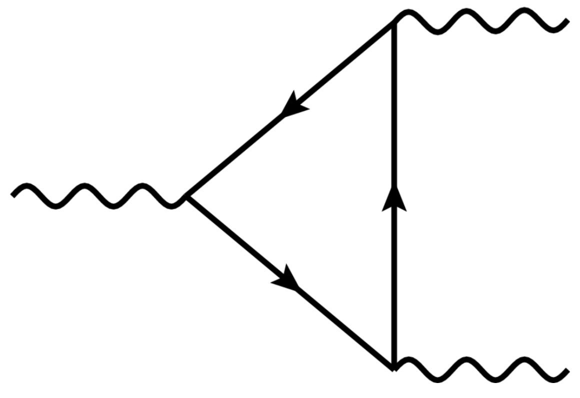

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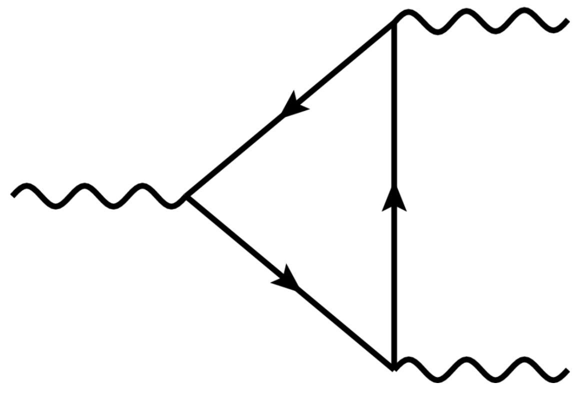

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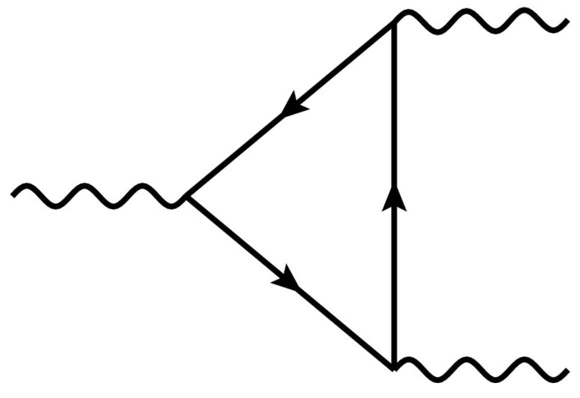

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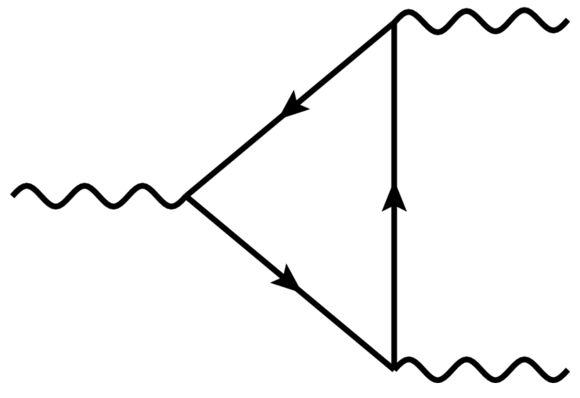

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$$\mathscr{D}(\Sigma_3) \supset \int [Da] \exp\left(\frac{2\pi i}{N} \int_{\Sigma_3} J_A^{(0)} + \frac{iN}{4\pi} \int_{\Sigma_3} a \wedge da + \frac{i}{2\pi} \int_{\Sigma_3} a \wedge dA \right), \qquad \mathscr{D}_{1/N}(\Sigma_3) \otimes \overline{\mathscr{D}}_{1/N}(\Sigma_3) = \mathscr{C}(\Sigma_3) \neq 1$$

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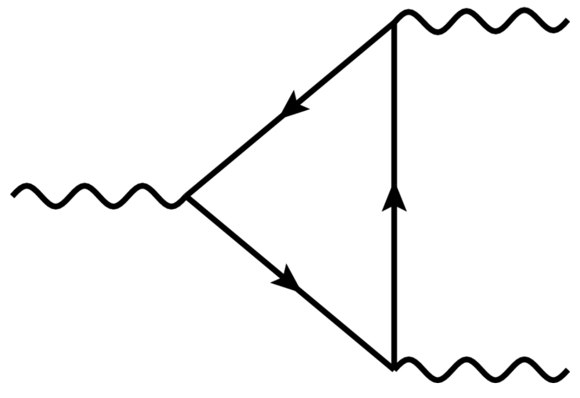

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Emergence constraint: breaking $U(1)_m^{(1)} \Rightarrow$ breaks also $\Gamma_A^{(0)}$, therefore $E_A \lesssim E_m$

Non-invertible sym in 4d axion-Maxwell

The lagragian is

[Choi, Lam, Shao 22; Yokokura 22; Del Zotto, Dell'Acqua, Garding 24; Sehayek, Craig 26]

$$\mathscr{L} = -\frac{1}{2g^2} F \wedge \star F - \frac{1}{2} \mathsf{f}^2 d\vartheta \wedge \star d\vartheta - \frac{K\vartheta}{8\pi^2} F \wedge F$$

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Goal: study more general symmetry structure in a large class of 4d models of vectors with a neutral sector, i.e. Gaillard-Zumino (GZ) models

Gaillard-Zumino Models

[Gaillard-Zumino '81]

Effective field theories in 4 space-time dimensions with Lagrangian

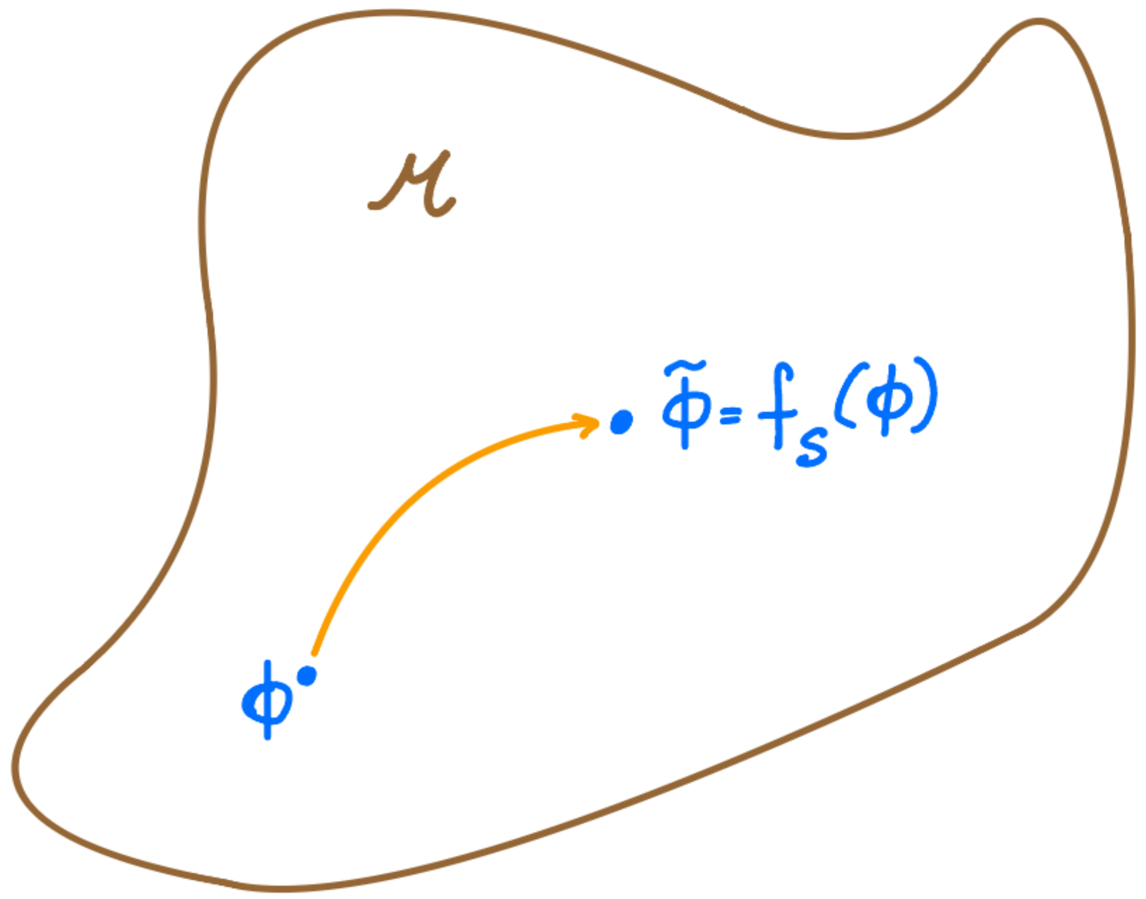
$$\mathscr{L}(F, \phi)$$

Field Content:

- Gauge fields: $F^I = dA^I, \quad I = 1, \ldots n$
- Neutral sector: ϕ^i scalars or fermions, we will usually consider scalars

These models have a classical invariance of the e.o.m. given by

- $\phi^i \to f_{\mathscr{S}}(\phi^i), \qquad \mathscr{S} \in \mathscr{G} \subset \mathrm{Sp}(2n, \mathbb{R})$
- $\begin{pmatrix} F^I \\ G_J \end{pmatrix} \to \mathscr{S} \cdot \begin{pmatrix} F^I \\ G_J \end{pmatrix}, \qquad G_J = 2\pi \frac{\partial \mathscr{L}}{\partial F^J}$
- $\mathscr{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, is a continuous symplectic rotation
- $\mathscr{G}$ is a continuous group, called GZ group, and $\mathscr{S}$ is an element of its symplectic representation



Gaillard-Zumino models examples

A first example that we encountered and fits into the GZ models is axion-Maxwell

A second example with $n = 1$ is axion-dilaton-Maxwell or τ-Maxwell, where $\tau = \frac{\vartheta}{2\pi} + ie^{-\phi}$

$$\mathscr{L} = -\frac{1}{4\pi}\mathrm{Im}\tau\, F \wedge \star F - \frac{1}{4\pi}\mathrm{Re}\tau\, F \wedge F - \mathsf{f}^2 \frac{d\tau \wedge \star d\bar{\tau}}{(\mathrm{Im}\tau)^2}$$

A third set of example with n $U(1)$ vector fields and $n-1$ complex scalars $\phi^i = a^i + is^i$ is the bosnic sector of N=2 supergravity in 4d,

$$\mathscr{L} = \frac{1}{4\pi}\mathrm{Im}\mathscr{N}_{IJ}(\phi)F^I \wedge \star F^J + \frac{1}{4\pi}\mathrm{Re}\mathscr{N}_{IJ}(\phi)F^I \wedge F^J - \frac{1}{2}\mathscr{K}_{ij}(\phi)d\phi^i \wedge \star d\phi^j$$

We will study the GZ symmetries of these models later on.

Gaillard-Zumino Condition

[Gaillard-Zumino '81]

The Lagrangian of the GZ model is such that,

$$\mathscr{L}(F, f_{\mathcal{S}}(\phi)) = \mathscr{L}_{\mathcal{S}}(F, \phi)$$

$\mathscr{L}_{\mathcal{S}}(F, \phi)$ describes a physically equivalent (dual) theories.

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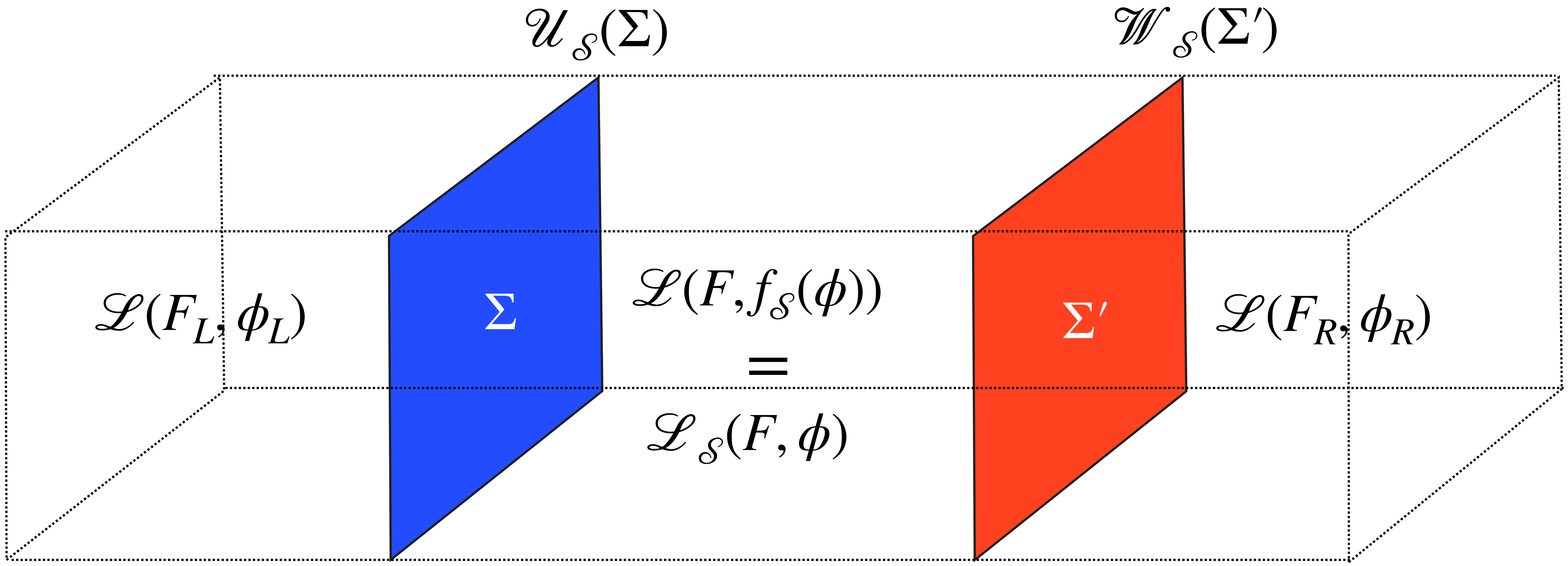
It tells us that the transformation on the neutral sector can be absorbed by a symplectic transformations on the field strengths

GZ non-invertible topological defects

Gaillard-Zumino topological defects

For any $\mathscr{S} \in \mathscr{G}_{\mathbb{Q}} \equiv \mathscr{G} \cap \mathrm{Sp}(2n, \mathbb{Q})$ we can combine the non-topological defect with the interface



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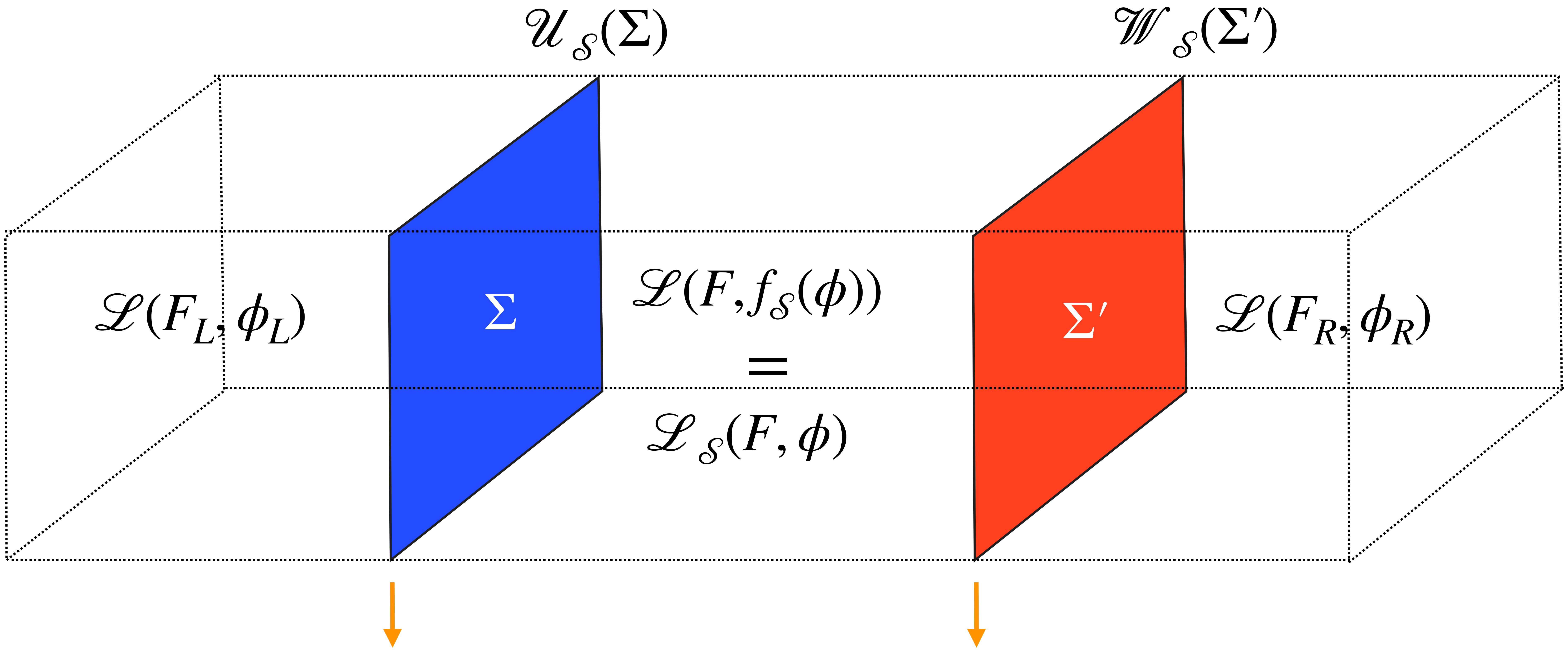

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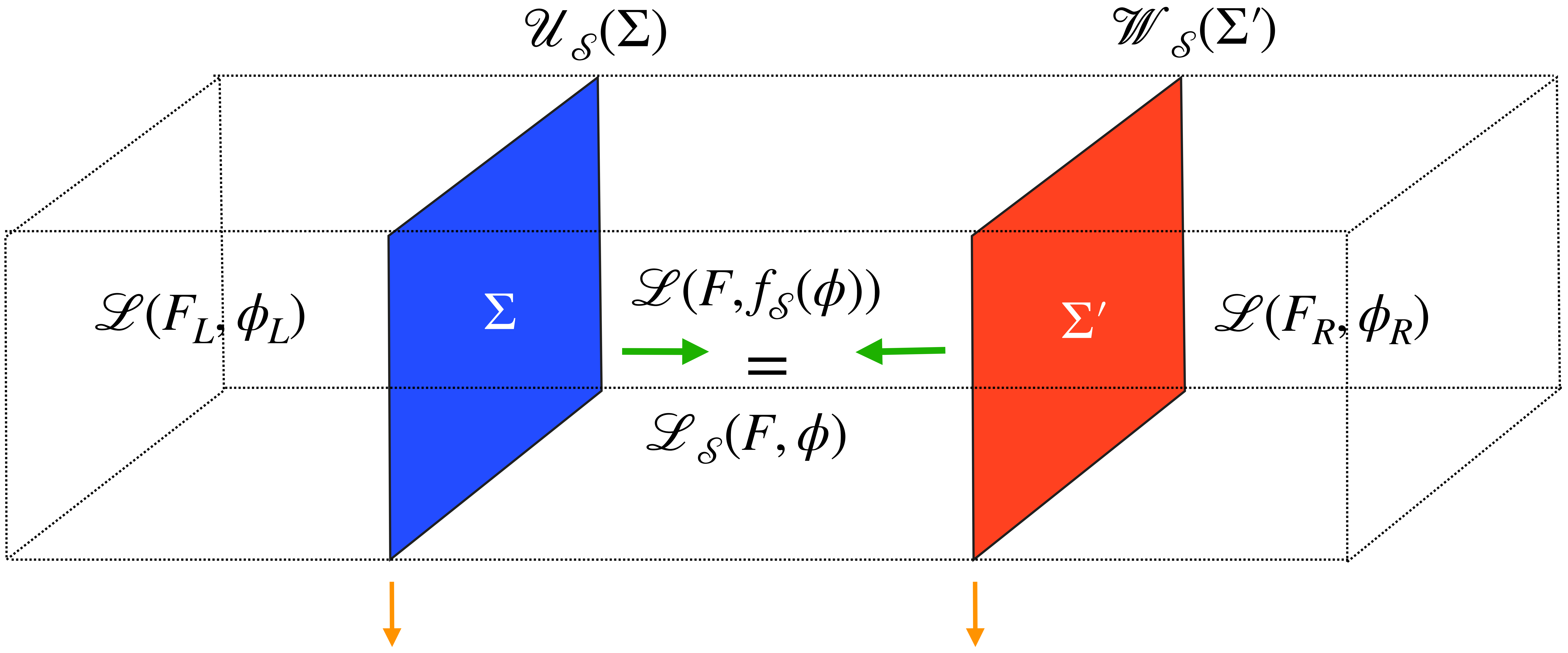

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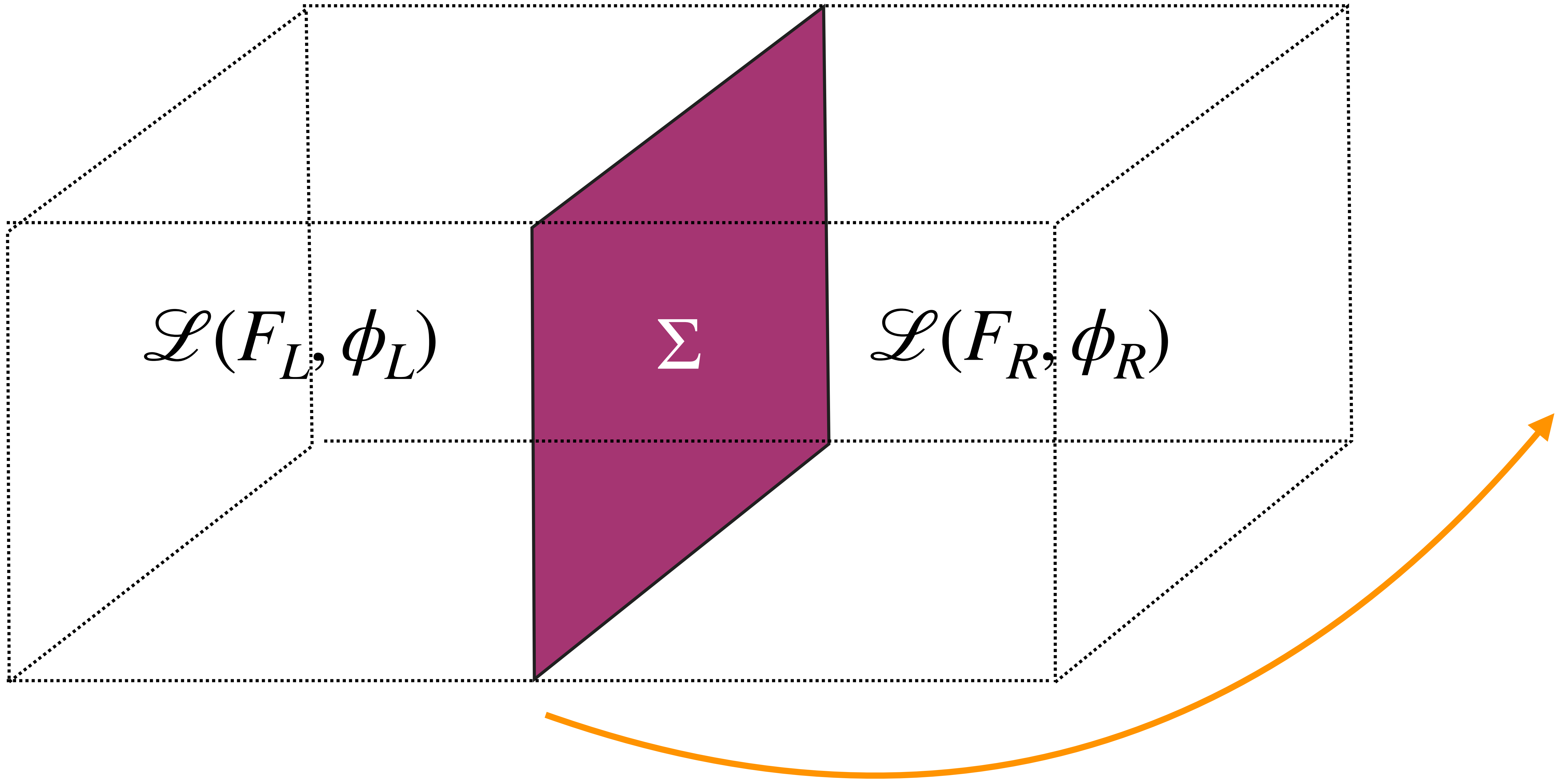
Gaillard-Zumino non-invertible defects

$$\mathscr{D}_{\mathcal{S}}(\Sigma) = \mathscr{U}_{\mathcal{S}}(\Sigma) \times \mathscr{W}_{\mathcal{S}}(\Sigma)$$



$$\begin{cases} \phi_L^i = f_{\mathcal{S}}^i(\phi_R), \\ \begin{pmatrix} F^I \\ G_J \end{pmatrix}_L = \mathcal{S} \cdot \begin{pmatrix} F^I \\ G_J \end{pmatrix}_R \end{cases}$$

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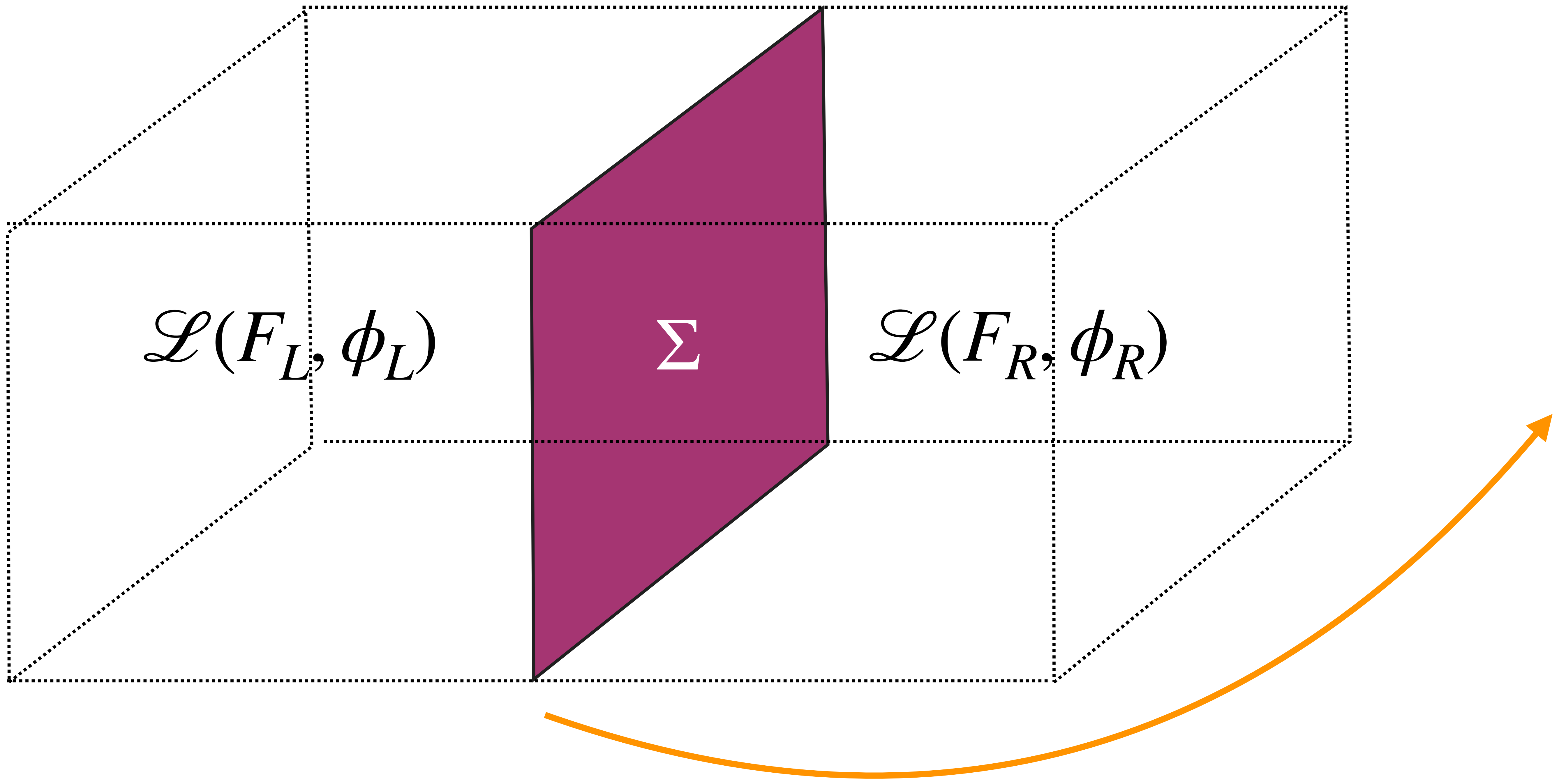

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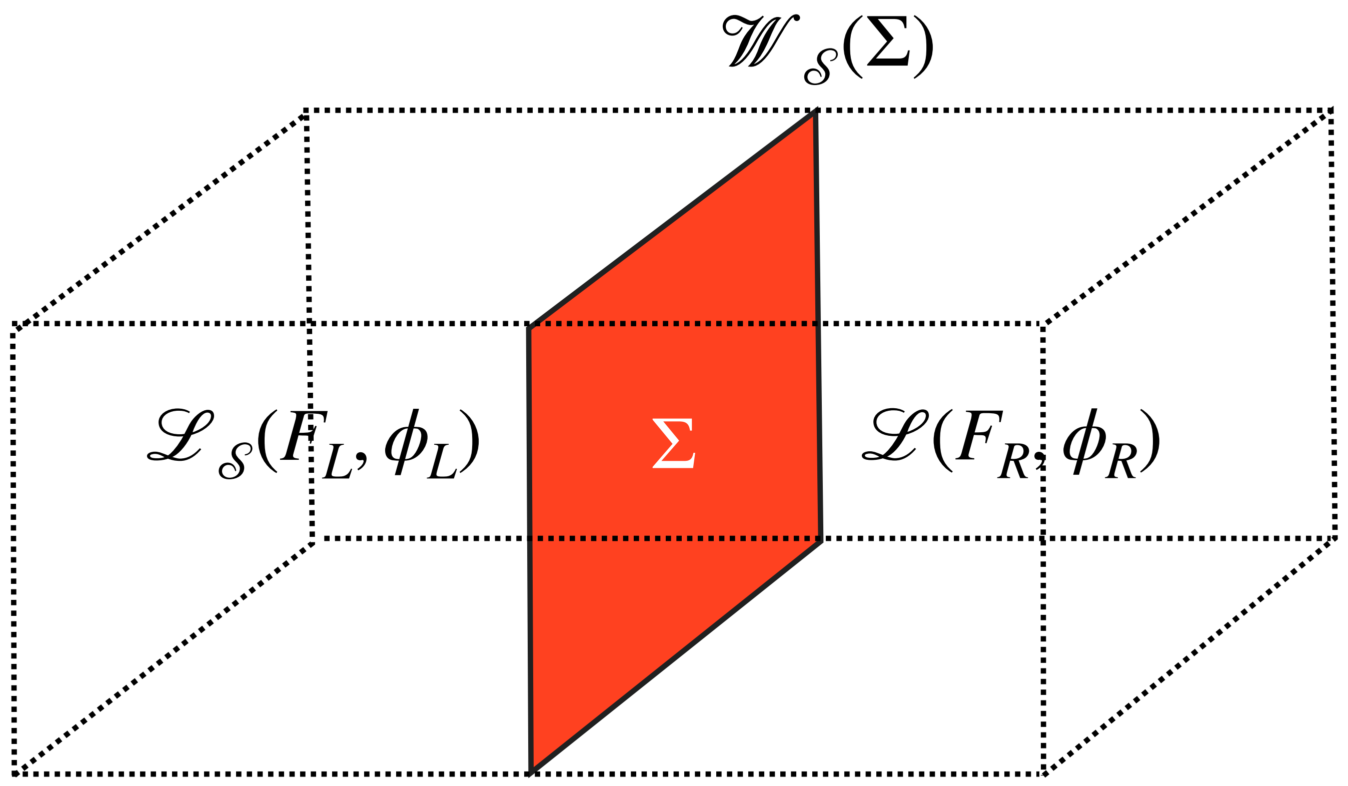
By combining the two operation we get topological symmetry defects,

- $\mathscr{G}_{\mathbb{Q}} \equiv \mathscr{G} \cap \mathrm{Sp}(2n, \mathbb{Q})$ label (non-invertible) symmetry defects, non-inv. follows from interfaces
- $\mathscr{G}_{\mathbb{Z}} \equiv \mathscr{G} \cap \mathrm{Sp}(2n, \mathbb{Z})$ are invertible

The GZ condition implies that the defect commute with $T_{\mu\nu}$, i.e. $T_{\mu\nu}(F_L, \phi_L) = T_{\mu\nu}(F_R, \phi_R)$

Gaillard-Zumino non-invertible interfaces



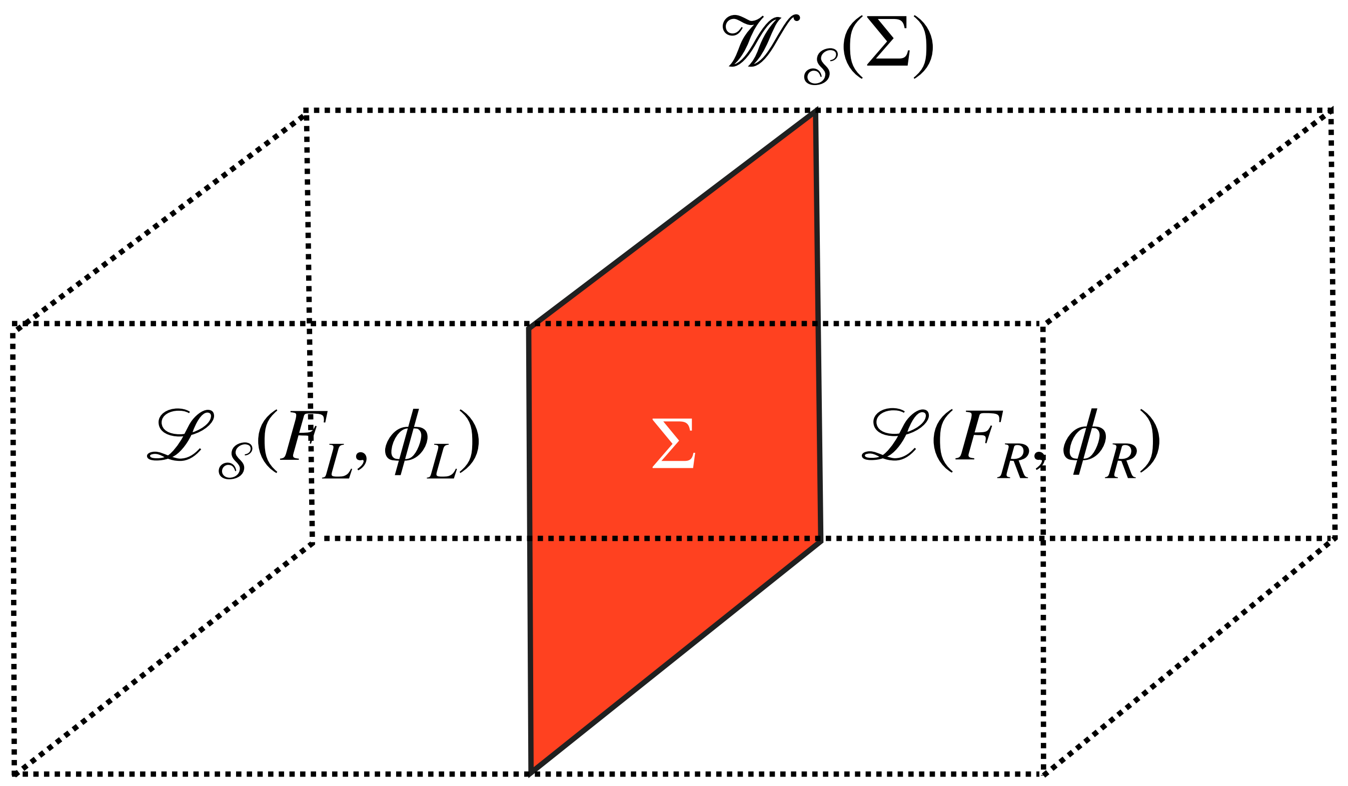
We can rely on the following decomposition on any $\mathcal{S} \in \mathrm{Sp}(2n, \mathbb{Q})$ (choice of generators)

$$\mathcal{S}_\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathcal{S}_\mathsf{A} = \begin{pmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{A}^{-1} \end{pmatrix} \quad \mathsf{A} \in \mathrm{GL}(n, \mathbb{Q}), \qquad \mathcal{S}_\mathsf{C} = \begin{pmatrix} 1 & 0 \\ \mathsf{C} & 0 \end{pmatrix}, \quad \mathsf{C}_{IJ} = \mathsf{C}_{JI} \in \mathbb{Q}$$

By fusing $\mathscr{W}_\mathsf{A}, \mathscr{W}_\mathsf{C}, \mathscr{W}_\Omega, \ \rightarrow \ \mathscr{W}_{\mathcal{S}}(\Sigma)$.

For Ω the interface has already been constructed and extensively studied, **S-duality wall**

Gaillard-Zumino non-invertible interfaces



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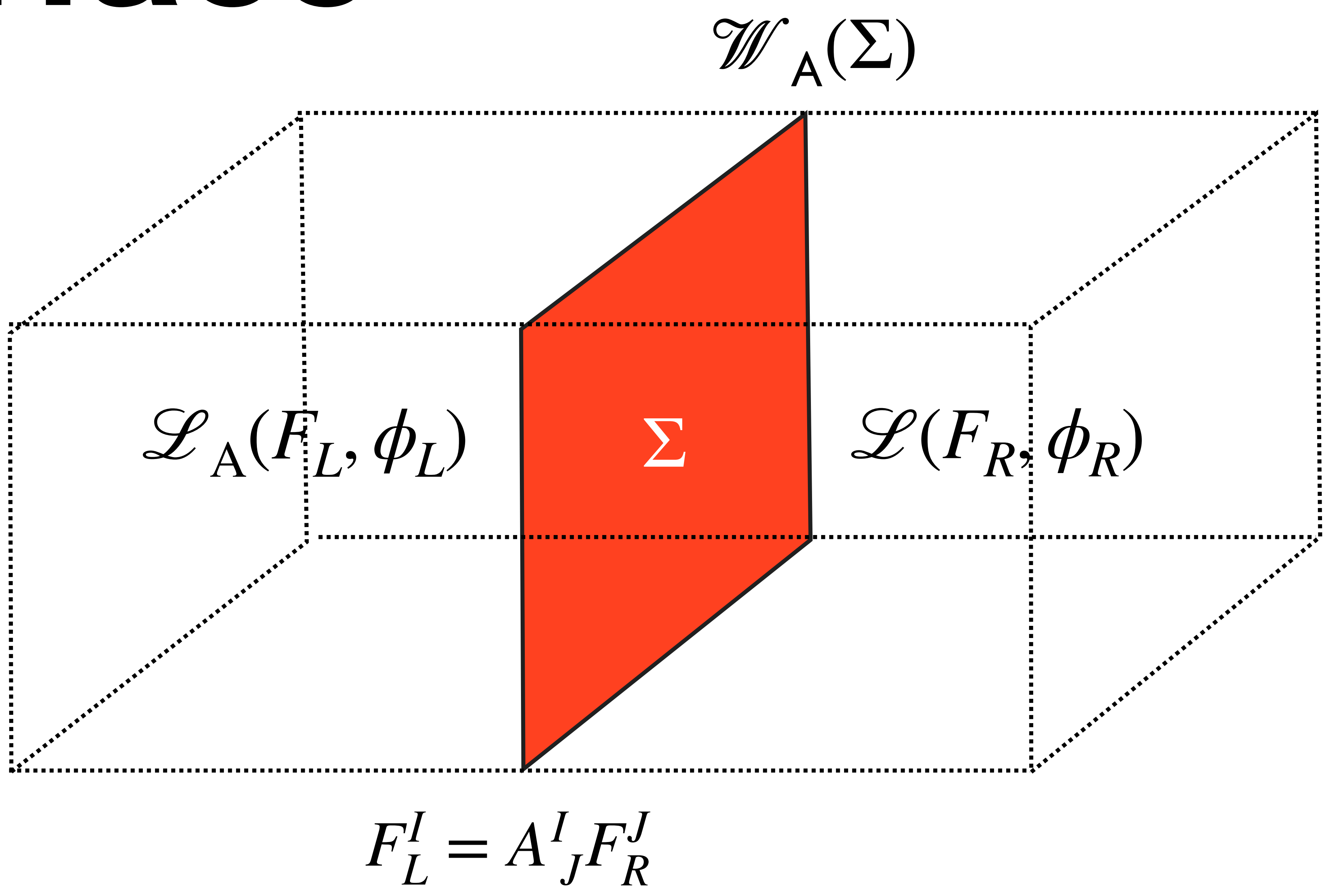
$$\mathscr{W}_{\Omega}(\Sigma) = \exp\left(\frac{i}{2\pi}\delta_{IJ}\oint_{\Sigma} A_L^I \wedge dA_R^J\right) \rightarrow \begin{pmatrix} F^I \\ G_J \end{pmatrix}_L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} F^I \\ G_J \end{pmatrix}_R$$

$\mathscr{W}_{\mathsf{A}}(\Sigma)$ (non-invertible) interface

Let us pick $\mathscr{S}_{\mathsf{A}} = \begin{pmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{A}^{-1} \end{pmatrix} \quad \mathsf{A} \in \mathsf{GL}(n, \mathbb{Q}),$



This interface is the generalization to the gauging interface of (non-anomalous) finite subgroup of the electric and magnetic 1-form symmetries.

We can consider a matrix factorization $\mathrm{A} = \mathrm{M}^{-1}\mathrm{E}, \quad \mathrm{M}, \mathrm{E} \in \mathrm{Mat}(n, \mathbb{Z})$

For n=1
[Córdova-Ohmori ’23
Niro-Roumpedakis ’22
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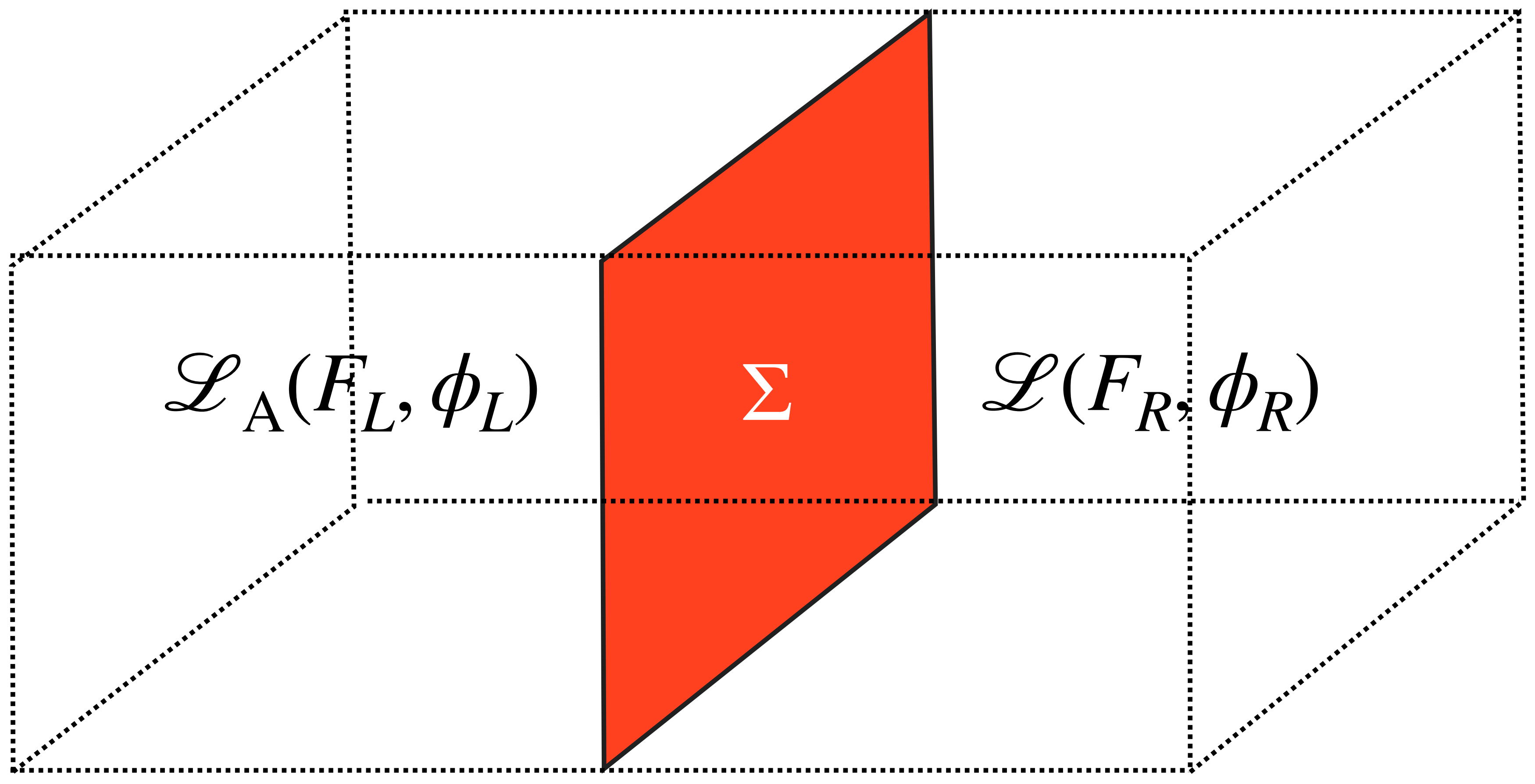

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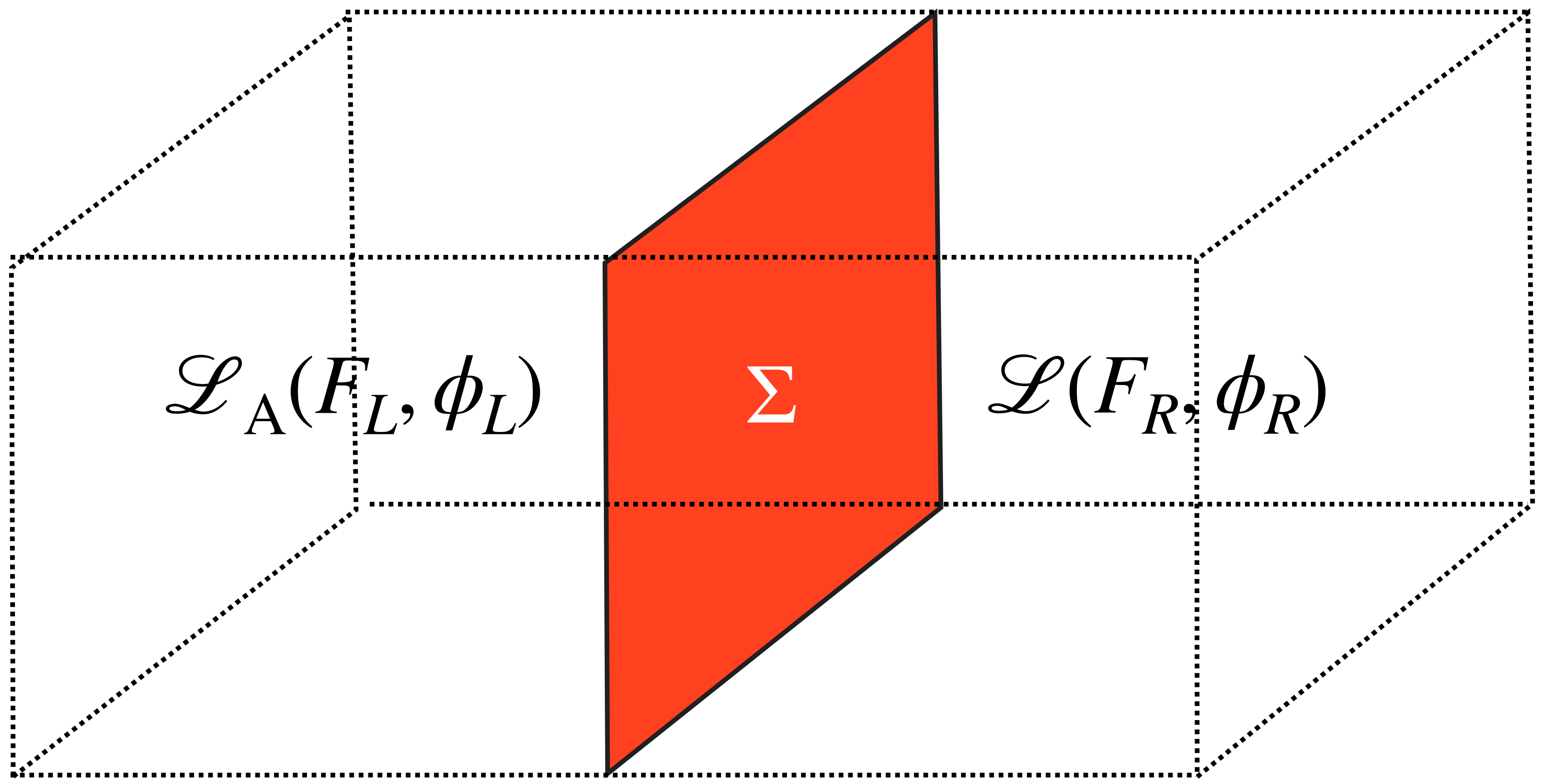

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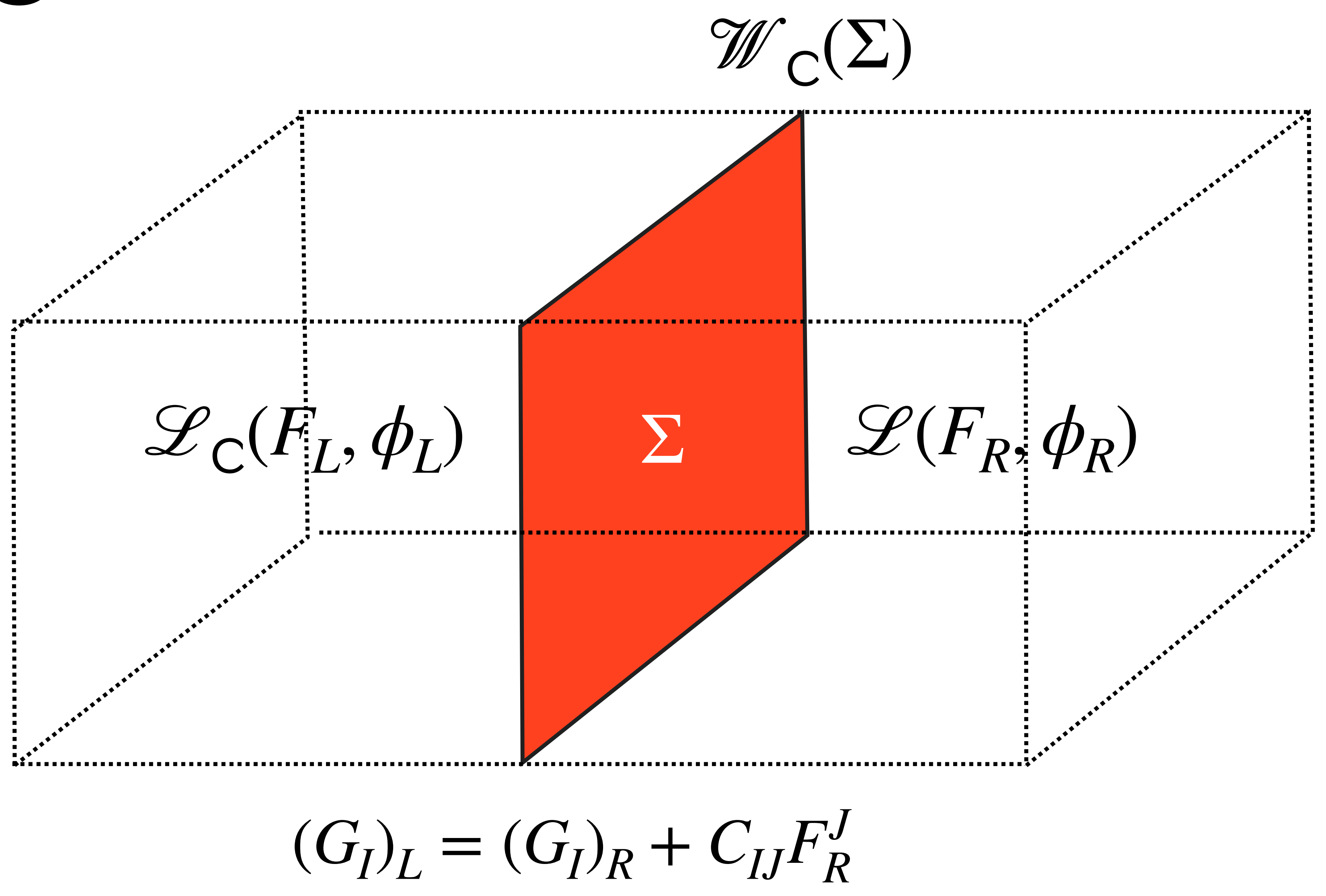
For $\mathsf{A} \in \mathrm{GL}(n, \mathbb{Z})$ the interface provides a rotation of the gauge fields (no rescaling, invertible)

$\mathscr{W}_{\mathsf{C}}(\Sigma)$ (non-invertible) interface

Let us pick $\mathscr{S}_{\mathsf{C}} = \begin{pmatrix} 1 & 0 \\ \mathsf{C} & 1 \end{pmatrix}$ with $\mathsf{C}_{IJ} = \mathsf{C}_{JI} \in \mathbb{Q}$

$$\mathscr{L}_{\mathsf{C}}(F_L, \phi_L) = \mathscr{L}(F_R, \phi_R) + \frac{1}{4\pi}\mathsf{C}_{IJ}F_R^I \wedge F_R^J$$

The two Lagrangian are equivalent only if $\mathsf{C}_{IJ} \in \mathbb{Z}$



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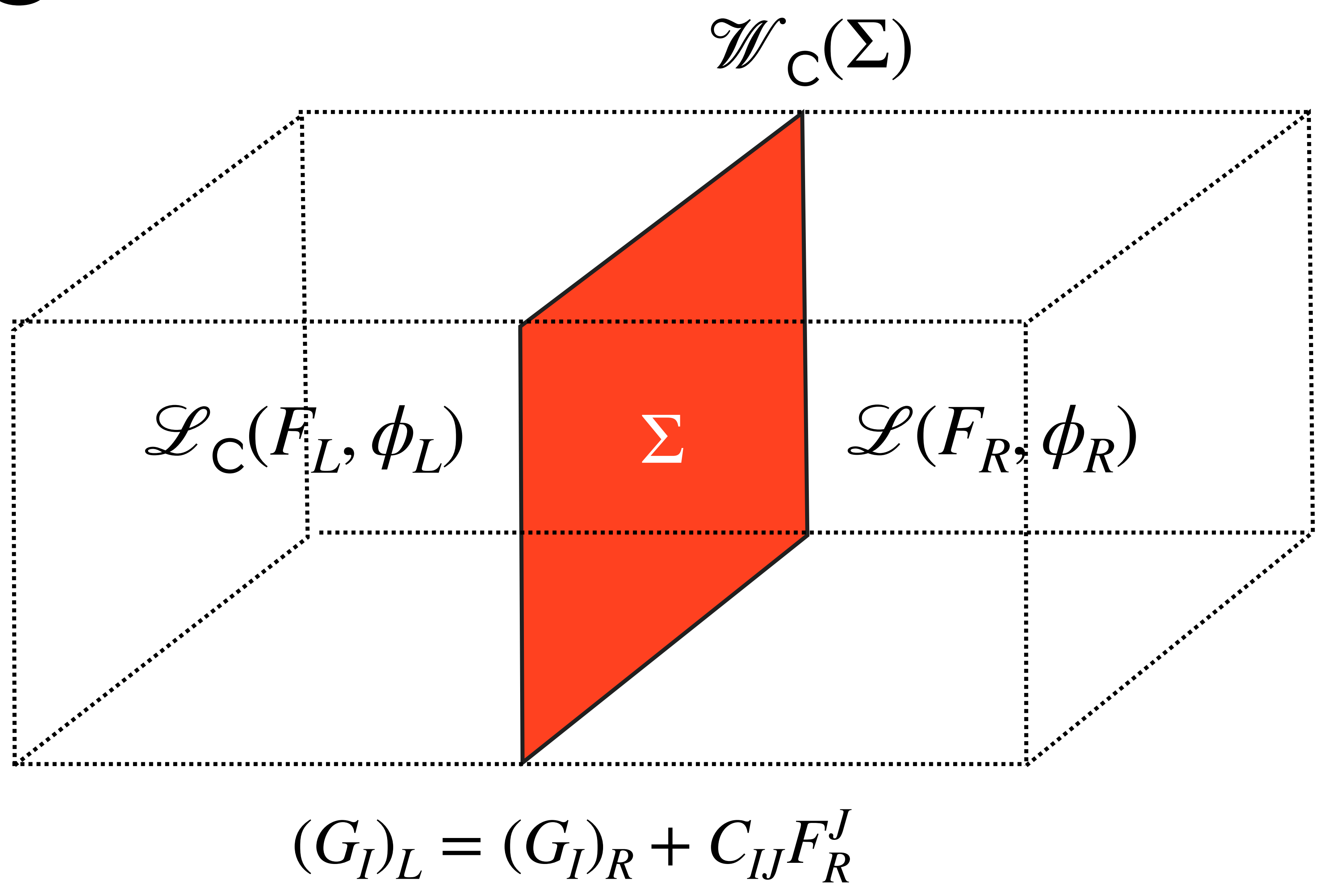

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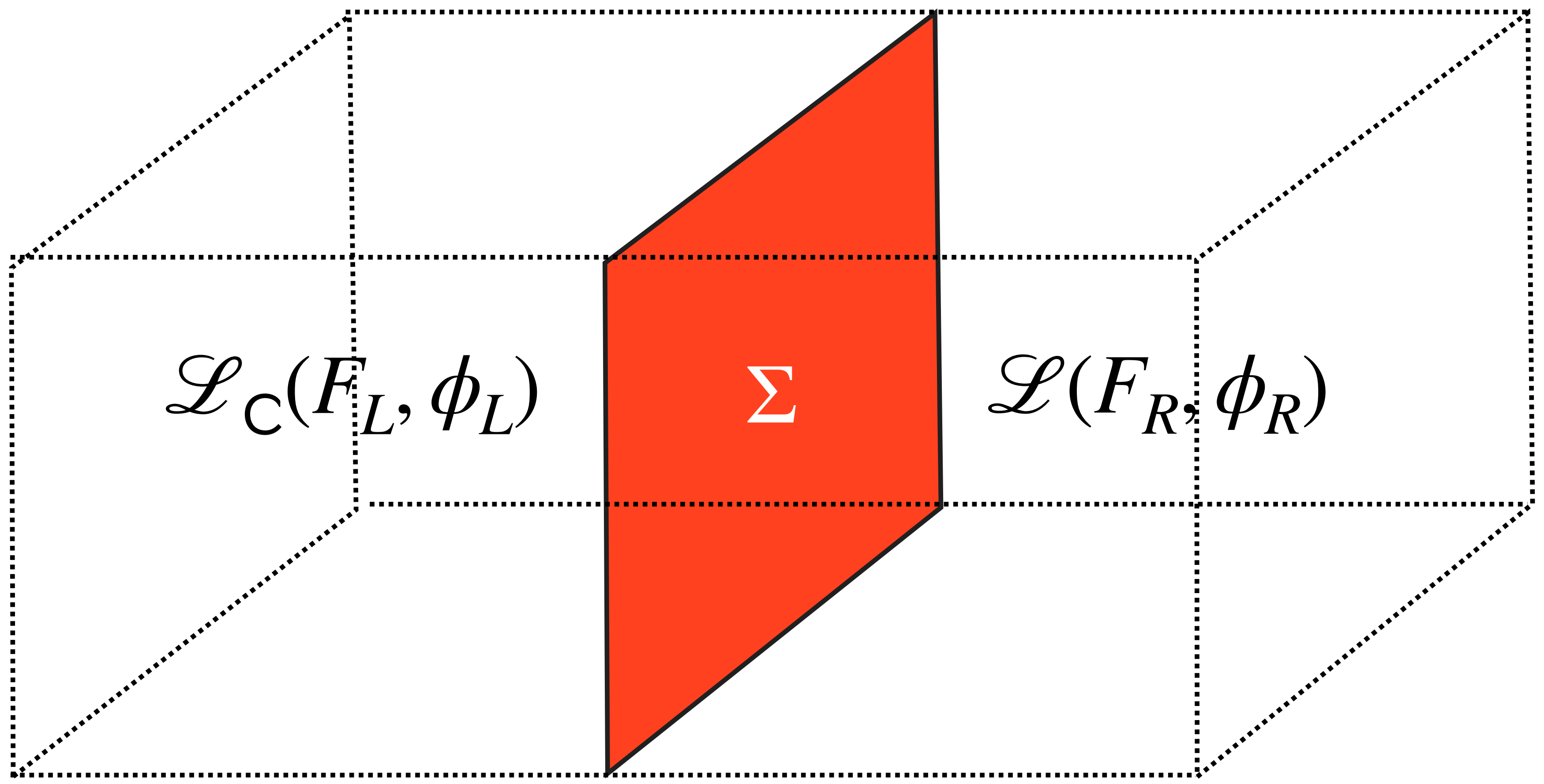

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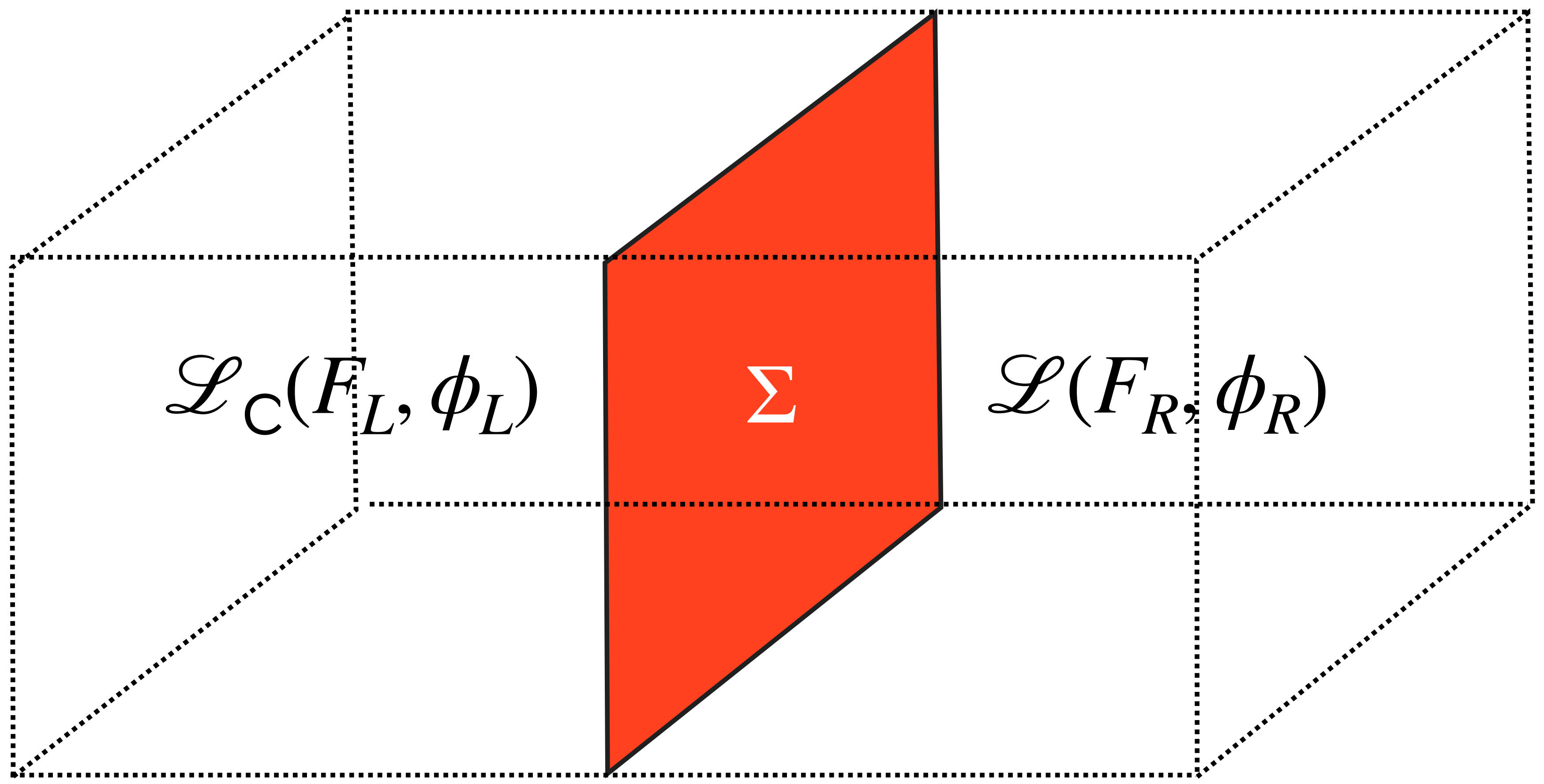

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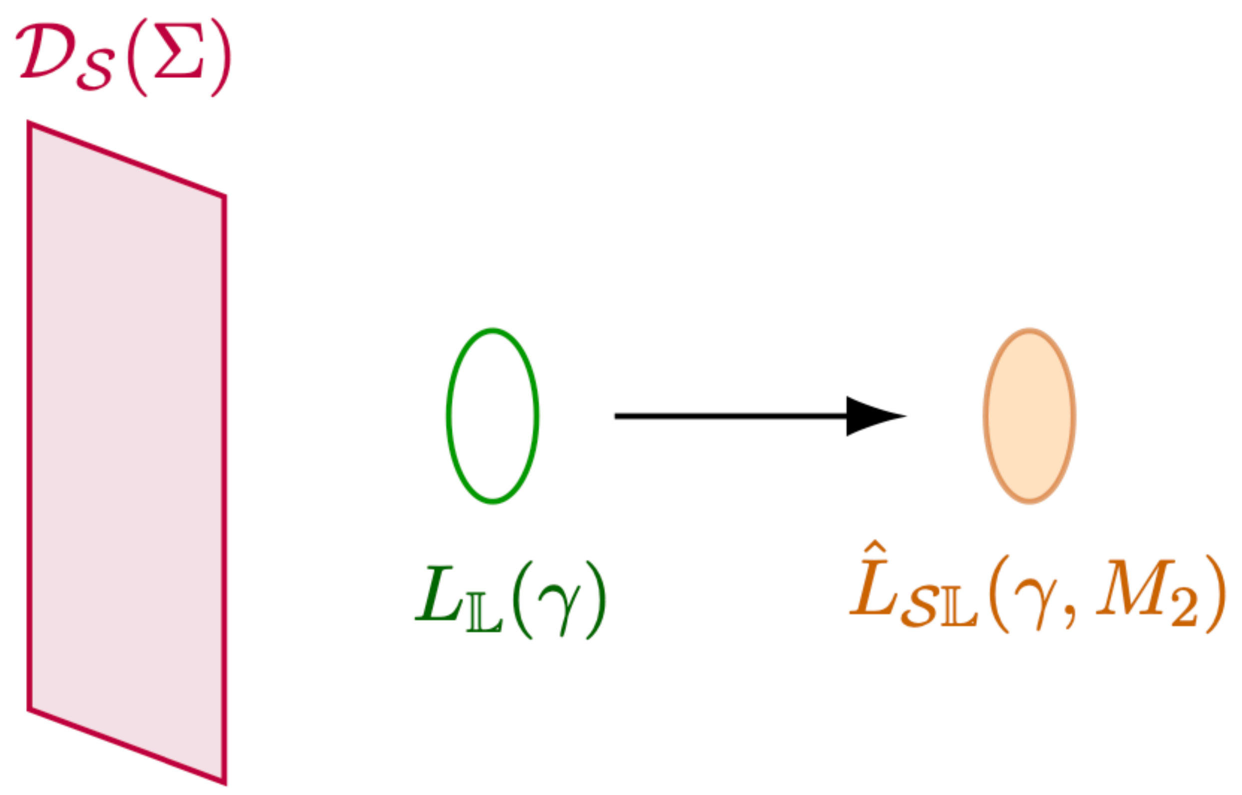
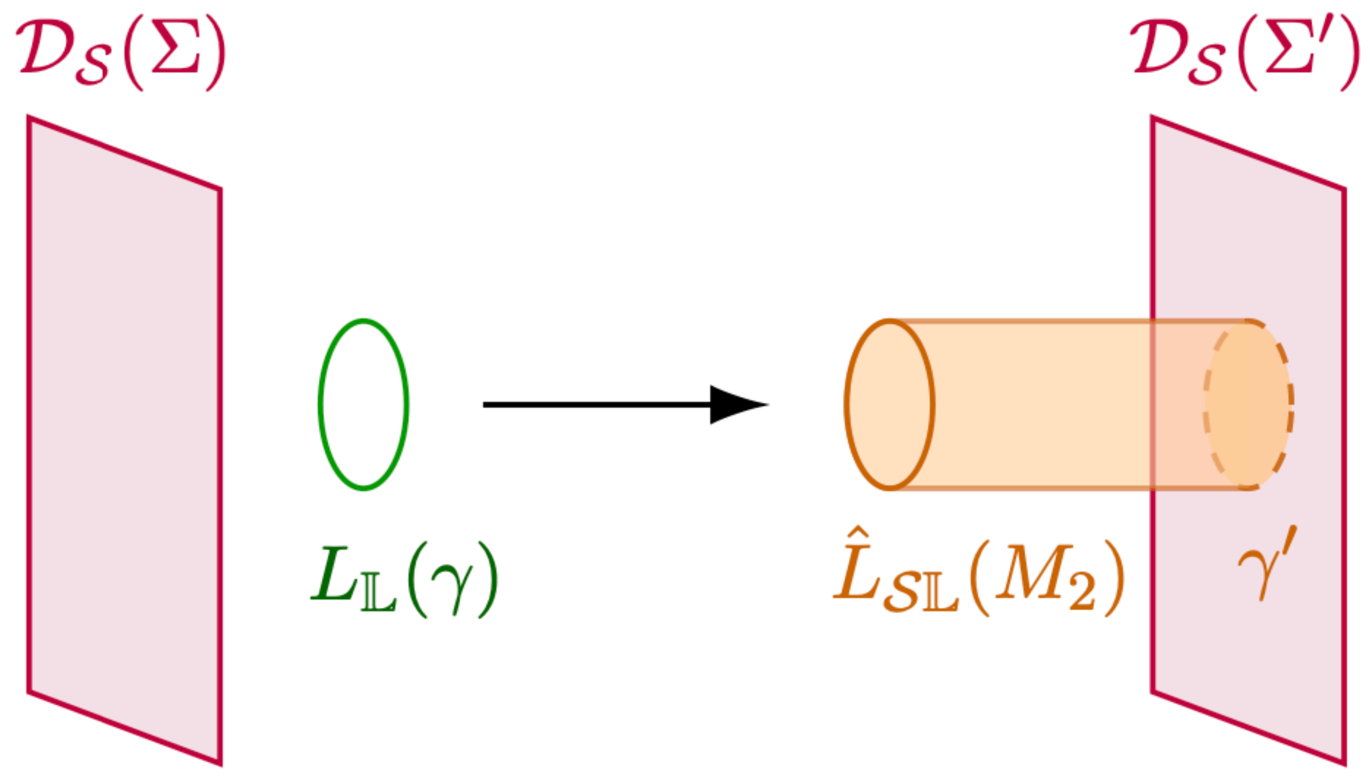
$\mathscr{W}_{\mathsf{A}}(\Sigma)$ and $\mathscr{W}_{\mathsf{C}}(\Sigma)$ can be obtained from half-space gauging [Choi-Cordova-Hsin-Lam-Shao '21-'22, Córdova-Ohmori '22] (n=1)

Non-invertibility

Action on lines $L_{\mathbb{L}}(\gamma) = \exp\left(i\tilde{\ell}\oint_\gamma \tilde{A} - i\ell\oint_\gamma A\right)$, $(\mathbb{L} = (\tilde{\ell}, \ell))$ (n=2) is dictated by $\mathcal{S} \in \mathcal{G} \cap \mathrm{Sp}(2,\mathbb{Q})$ and interfaces gluing conditions. It is non-invertible, genuine lines become non genuine, if $\mathcal{S}\mathbb{L} \neq \mathbb{Z}^2$





To recap:

- non-invertible topological defects generically labelled by $\mathcal{G}_{\mathbb{Q}}$ elements

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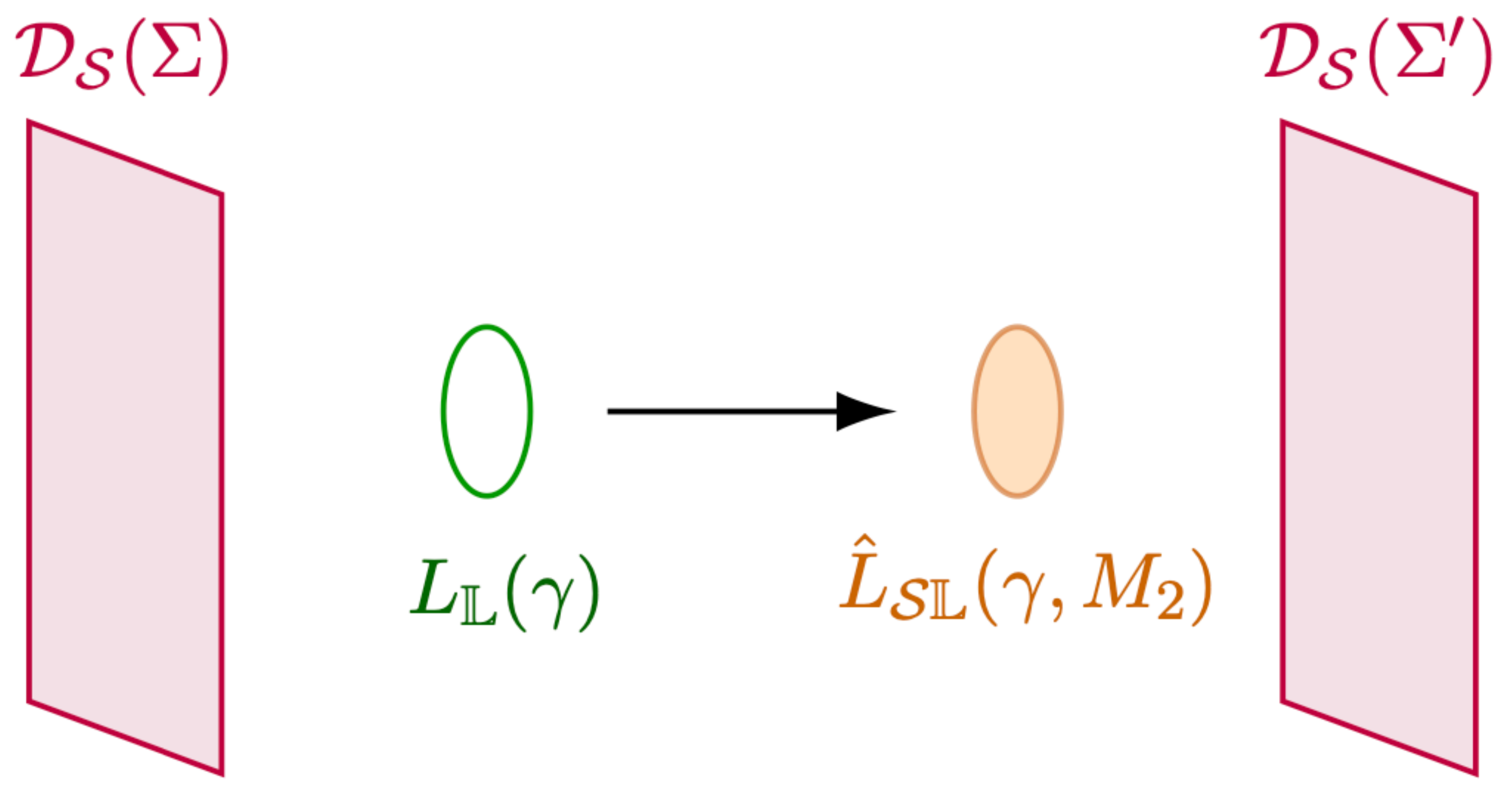

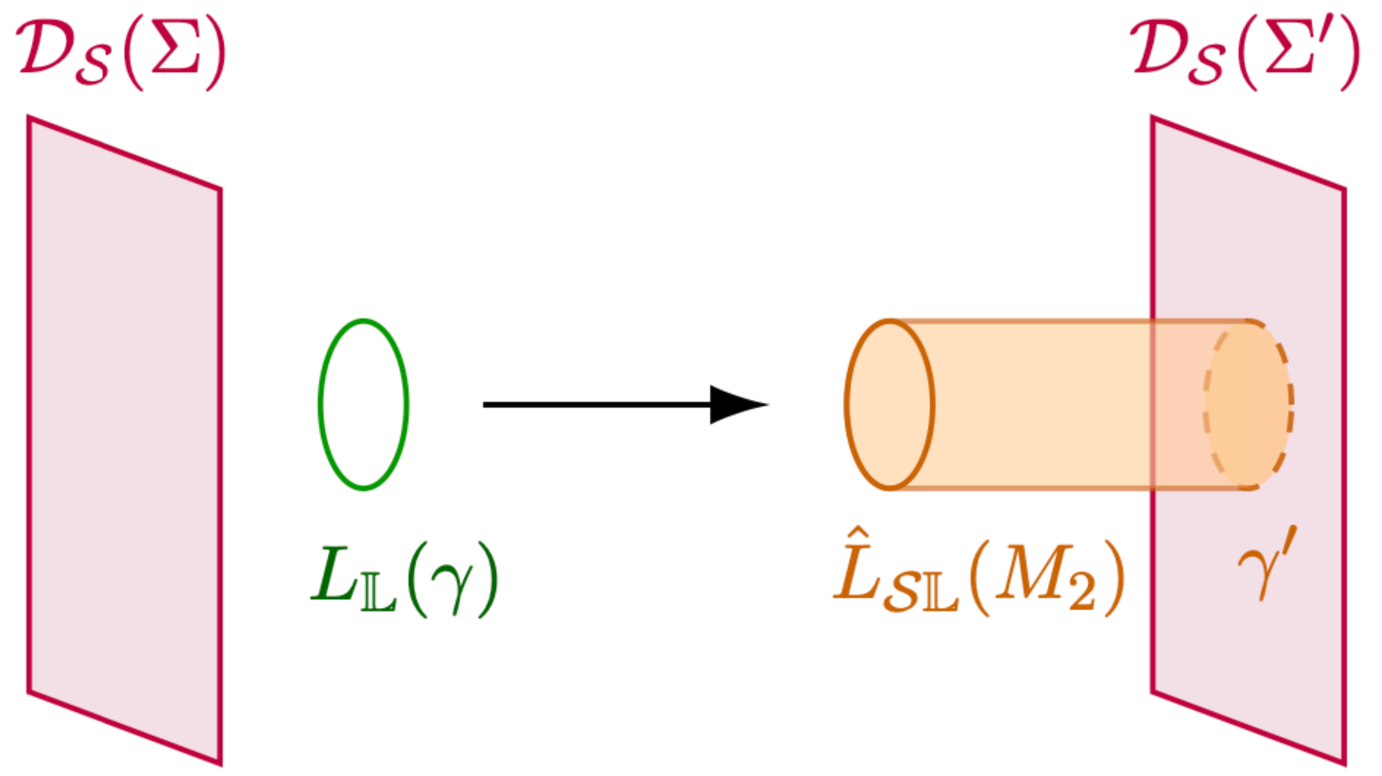

To recap:

- non-invertible topological defects generically labelled by $\mathcal{G}_{\mathbb{Q}}$ elements
- invertible topological defects, $\mathcal{G}_{\mathbb{Z}}$

axion-dilaton Maxwell

τ- Maxwell and Sl(2,R)

A simple $n = 1$ generalization is axion-dilaton-Maxwell or τ-Maxwell, where $\tau = \frac{\vartheta}{2\pi} + ie^{-\phi}$

$$\mathscr{L} = -\frac{1}{4\pi}\mathrm{Im}\tau\, F \wedge \star F - \frac{1}{4\pi}\mathrm{Re}\tau\, F \wedge F - \mathsf{f}^2\,\frac{d\tau \wedge \star d\bar{\tau}}{(\mathrm{Im}\tau)^2}$$

As a GZ model has a and $\mathcal{S} \in \mathcal{G} = \mathrm{Sp}(2,\mathbb{R}) = \mathrm{Sl}(2,\mathbb{R})$, the action on τ reads

$$f_{\mathcal{S}}(\tau) = \frac{\mathsf{d}\tau - \mathsf{c}}{\mathsf{a} - \mathsf{b}\tau}, \qquad \mathsf{ad} - \mathsf{bc} = 1$$

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Due to the interfaces we focus on the rational part $\mathrm{Sl}(2,\mathbb{Q})$, it is convenient to pick a set of generators

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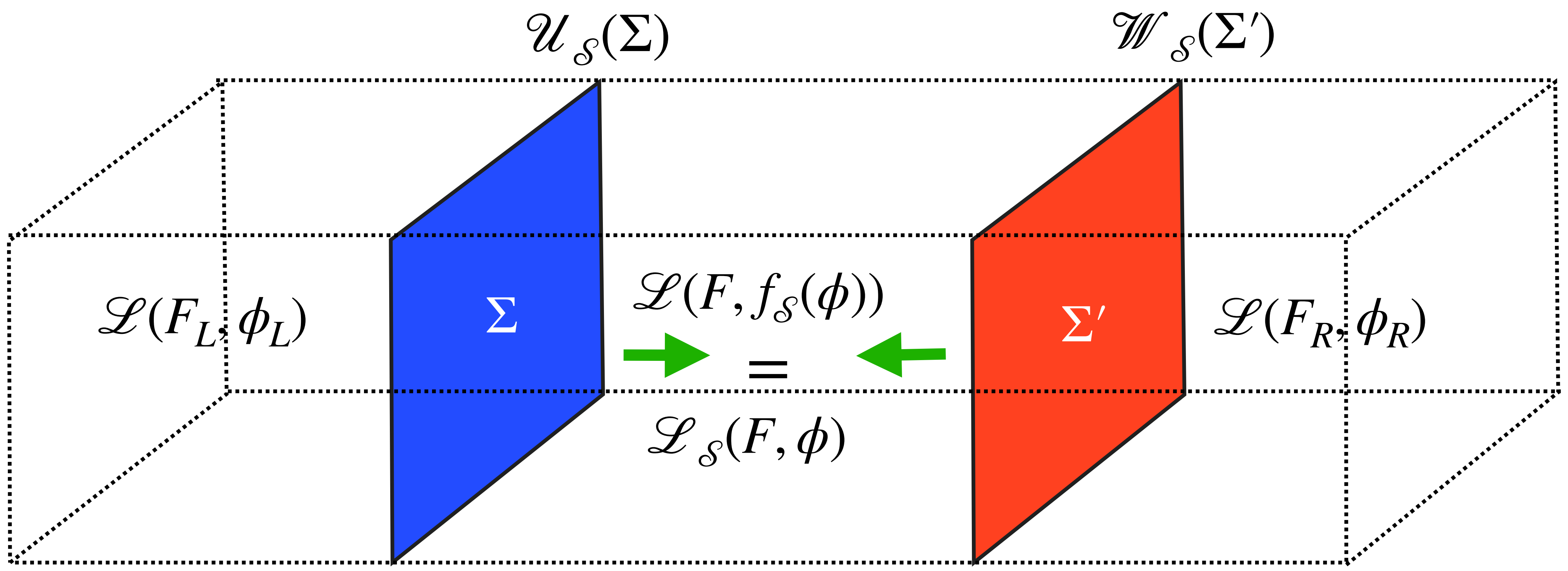
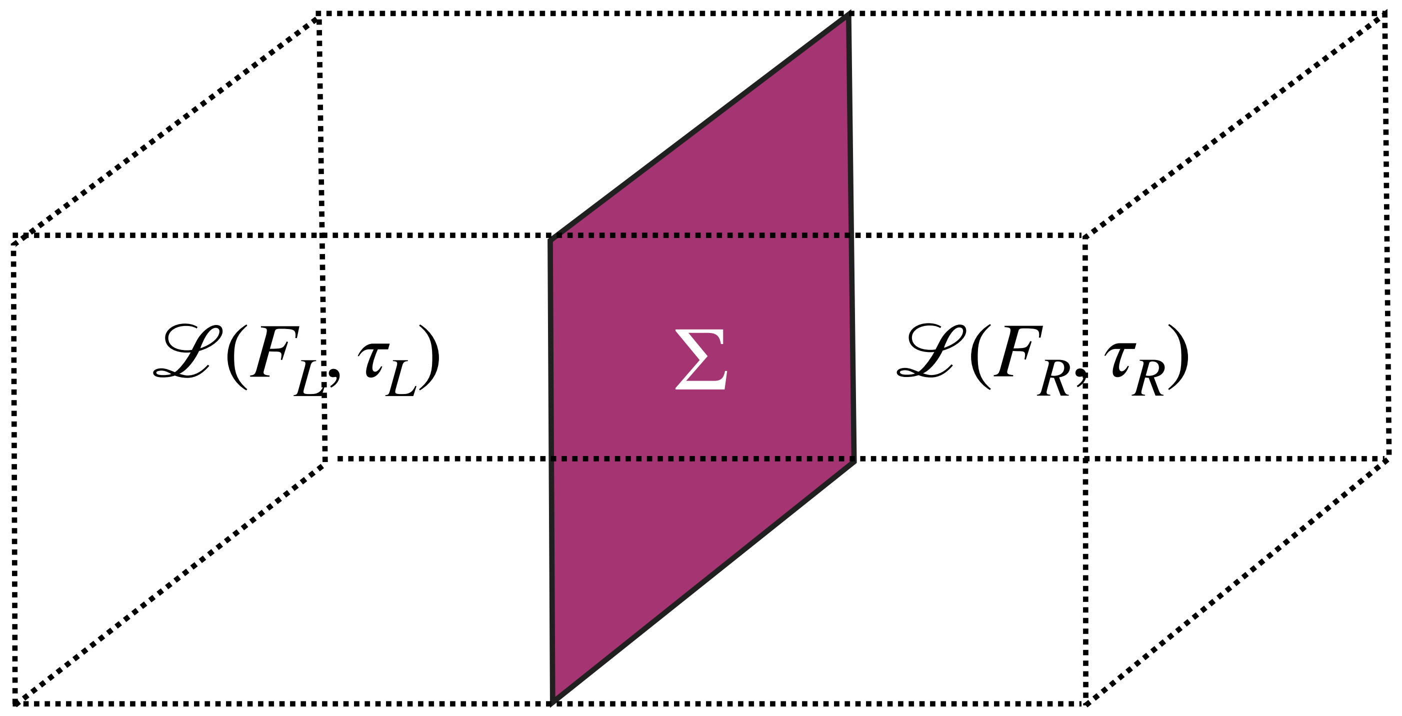
$\mathsf{p}, \mathsf{n}, \mathsf{e}, \mathsf{m} \in \mathbb{Z}$

GZ Topological defects for τ- Maxwell

We apply the same strategy where $\mathcal{G} = \mathrm{Sl}(2,\mathbb{R})$





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$\mathcal{U}_{\mathcal{S}}(\Sigma) = \exp\left(\frac{2\pi i p}{n}\oint_\Sigma \star j_T\right)$, $\quad \star j_T = -\mathsf{f}^2\,\frac{\xi_T(\tau)\star d\bar{\tau} + \bar{\xi}_T(\bar{\tau})\star d\tau}{(\mathrm{Im}\tau)^2}$, $\quad T = \begin{pmatrix} \mathsf{u} & \mathsf{v} \\ \mathsf{w} & -\mathsf{u}\end{pmatrix} \in \mathrm{sp}(2,\mathbb{R})\,.$ $\quad \delta_T\tau = \xi_T(\tau) = -\mathsf{w} - 2\mathsf{u}\tau + \mathsf{v}\tau^2$

For the interface we specify the general ones for n=2, and we obtain $\mathcal{W}_{\mathrm{A}}, \mathcal{W}_{\mathrm{C}}, \mathcal{W}_{\Omega}, \ \rightarrow \ \mathcal{W}_{\mathcal{S}}(\Sigma)$.

$\mathcal{D}_{\mathcal{S}}(\Sigma) = \mathcal{U}_{\mathcal{S}}(\Sigma) \times \mathcal{W}_{\mathcal{S}}(\Sigma)$ define the symmetry defects labeled by $\mathcal{G}_{\mathbb{Q}} = \mathrm{Sl}(2,\mathbb{Q})$

N=2 SUGRA

The models have n $U(1)$ vector fields and $n-1$ complex scalars $\phi^i = a^i + is^i$ and Lagrangian

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$$\mathscr{L} = \frac{1}{4\pi}\mathrm{Im}\mathscr{N}_{IJ}(\phi)F^I \wedge \star F^J + \frac{1}{4\pi}\mathrm{Re}\mathscr{N}_{IJ}(\phi)F^I \wedge F^J - \frac{1}{2}\mathscr{K}_{ij}(\phi)d\phi^i \wedge \star d\phi^j$$

Axion-shift symmetry $\phi^i \to \phi^i + \alpha^i, \quad \alpha^i \in \mathbb{R}$

$\begin{pmatrix} F^I \\ G_J \end{pmatrix}$ transform with $\mathscr{S} = \begin{pmatrix} \mathsf{A}(\alpha) & \mathsf{B}(\alpha) \\ \mathsf{C}(\alpha) & \mathsf{D}(\alpha) \end{pmatrix} \in \mathbb{R}^{n-1} \subset \mathrm{Sp}(2n, \mathbb{R})$

We can construct sensible interfaces for $\mathbb{Q}^{n-1} = \mathbb{R}^{n-1} \cap \mathrm{Sp}(2n, \mathbb{Q})$,

$$\alpha^i = \alpha^i_{\mathbf{q},\mathbf{p}} \equiv \frac{q^i}{p^i} \in \mathbb{Q}, \qquad q^i, p^i \in \mathbb{Z},\ \gcd(q^i, p^i) = 1$$

The non-invertible defects are $\mathscr{D}_{\mathbf{q},\mathbf{p}}(\Sigma) = \mathscr{U}_{\mathbf{q},\mathbf{p}}(\Sigma) \times \mathscr{W}_{\mathbf{q},\mathbf{p}}(\Sigma)$ where $\mathscr{W}_{\mathbf{q},\mathbf{p}} = \mathscr{W}^{\mathbf{q},\mathbf{p}}_{\tilde{\mathsf{C}}} \times \mathscr{W}^{\mathbf{q},\mathbf{p}}_{\mathsf{A}}$

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Conclusions

Conclusions and outlook

We built non-invertible topological defects in general GZ models (Field content: gauge fields with neutral sector) associated to $\mathcal{G}_{\mathbb{Q}} \equiv \mathcal{G} \cap \mathrm{Sp}(2n, \mathbb{Q})$, and we discussed some explicit examples

In the last example $\mathcal{D}_{\mathbf{q},\mathbf{p}}(\Sigma)$ are labelled by $\mathcal{G}_{\mathbb{Q}} = \mathbb{Q}^{n-1}$. The axion are not periodic yet, to do this we need to gauge the invertible part $\mathcal{G}_{\mathbb{Z}} = \mathbb{Z}^{n-1}$, generated by $\mathcal{D}_{\mathbf{m}}(\Sigma), \quad \mathbf{m} \in \mathbb{Z}^{n-1}$

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Rich symmetry structure: emergent constraints, hierarchy of scales and charged defects (generalization of axion strings, monopoles) from symmetry structure vs $T_{F1} \lesssim T_{D4}$ and T_{NS5}
[Marchesano, Melotti 22; Marchesano, Wiesner 22; Hassled, Monnee, Weigand, Weisner 25]

Conclusions and outlook

- ’t Hooft anomalies and mixed gravitational anomalies
  - invertible part: $\mathscr{G}_{\mathbb{Z}}$ [Witten ’95] [Seiberg-Tachikawa-Yonekura ’18] [Hsieh-Tachikawa-Yonekura ’19]
  - non-invertible part labelled by $\mathscr{G}_{\mathbb{Q}}$ [Antinucci-Benini-Copetti-Galati-Rizi ‘23] [Cordova-Hsin-Zhang ‘23] [Antinucci-Copetti-Gai-Schäfer-Nameki ’25] [Del Zotto-Dell’Acqua-Garding ’25]
- Higher supersymmetry and breaking by gauging the discrete invertible part? [Delgado-van de Heisteeg-Raman-Torres-Vafa-Xu ’24]
- Toroidal compactifications lead to non-invertible $O(d, d, \mathbb{Q})$ symmetry. Laboratory to quantify the scale at which the non-invertible symmetries become approximate. [WIP with Bachas-Leone-Mancani-Martucci]
- Wormholes, worldsheet instantantons and GZ symmetries [Martucci, Risso, Valenti, Vecchi 24; Maldacena, Maloney McPeak 26; Di Ubaldo, Iliesiu, Lin, Yan 26]

Thank You

Backup slides

Fusion and non-invertibility

$\mathscr{D}_{\Omega}$ is invertible: $\mathscr{D}_{\Omega} \times \mathscr{D}_{\Omega^{-1}} = 1$, in addition many fusion rules will behave group theoretically

$$\mathscr{D}_{\mathcal{S}} \times \mathscr{D}_{\mathcal{S}'} = \mathscr{D}_{\mathcal{S}\mathcal{S}'}$$

$\mathscr{D}_{\mathsf{A}}$ is instead non-invertible and its fusion leads to the condensation defect

$$\mathscr{D}_{\mathsf{A}}^{(\mathsf{e},\mathsf{m})} \times \overline{\mathscr{D}}_{\mathsf{A}}^{(\mathsf{e},\mathsf{m})} = \mathscr{W}_{\mathsf{A}}^{(\mathsf{e},\mathsf{m})} \times \mathscr{W}_{\mathsf{A}^{-1}}^{(\mathsf{e},\mathsf{m})} = \mathscr{C}_{\mathsf{A}}^{(\mathsf{e},\mathsf{m})} \neq 1$$

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[Córdova-Ohmori '23
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Gaillard-Zumino defects

The construction proceeds in two steps. We first construct the (non-topological) defects implementing the transformation on the scalar, for $\mathscr{S} \in \mathscr{G} \subset \mathrm{Sp}(2n, \mathbb{R})$

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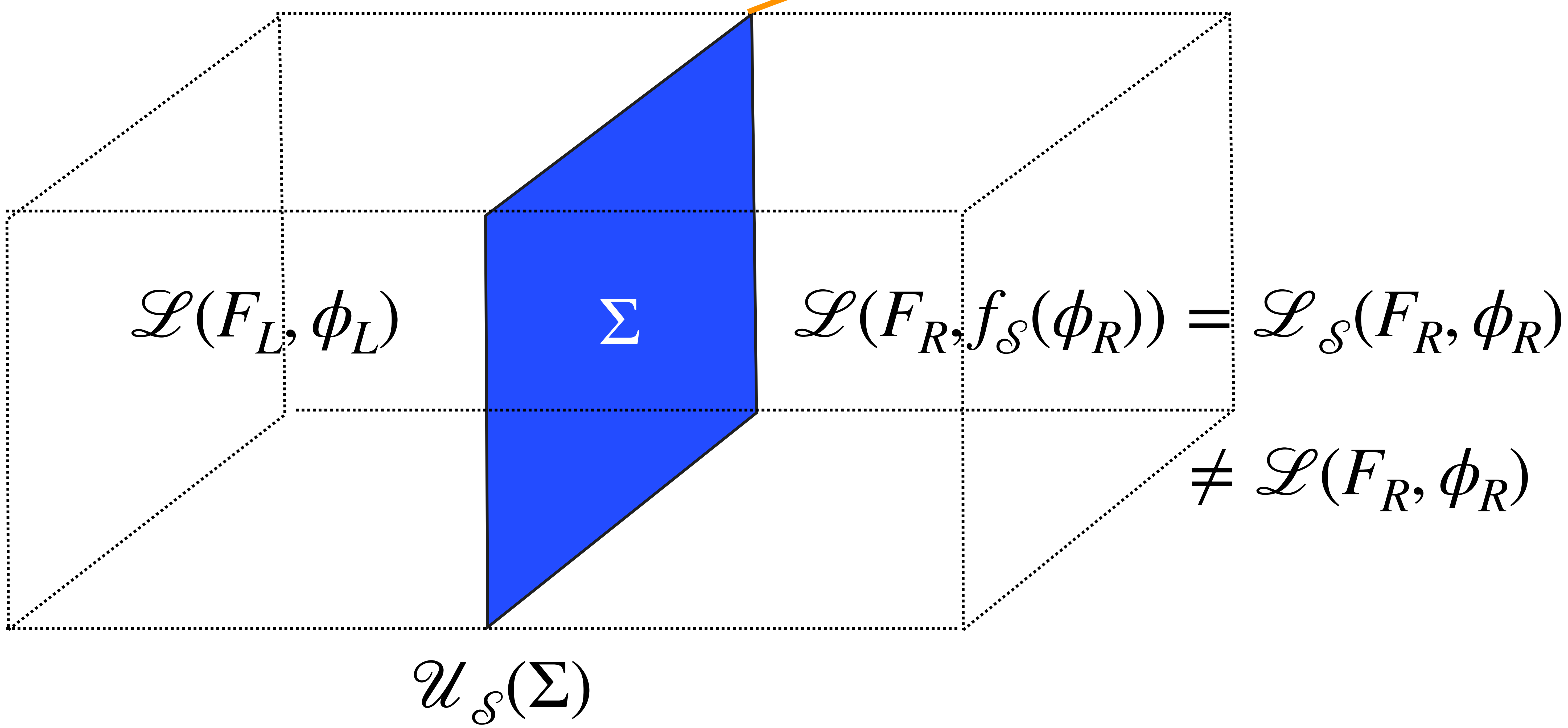

Gluing Condition:

$$\phi_L^i = f_{\mathscr{S}}^i(\phi_R)$$

GZ Topological defects

Non-invertible defect $\mathscr{D}_{\mathsf{A}}$ labelled by the element $\mathscr{S}_{\mathsf{A}}$ (rescaling or gauging):

[Córdova-Ohmori '23
Niro-Roumpedakis '22
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$$\mathscr{U}^{(\mathsf{e},\mathsf{m})}_{\mathsf{A}}(\Sigma) = \exp\left[4if^2 \log\left(\frac{\mathsf{m}}{\mathsf{e}}\right)\oint_\Sigma \frac{\mathrm{Re}\left(\bar{\tau} \star d\tau\right)}{(\mathrm{Im}\tau)^2}\right]$$

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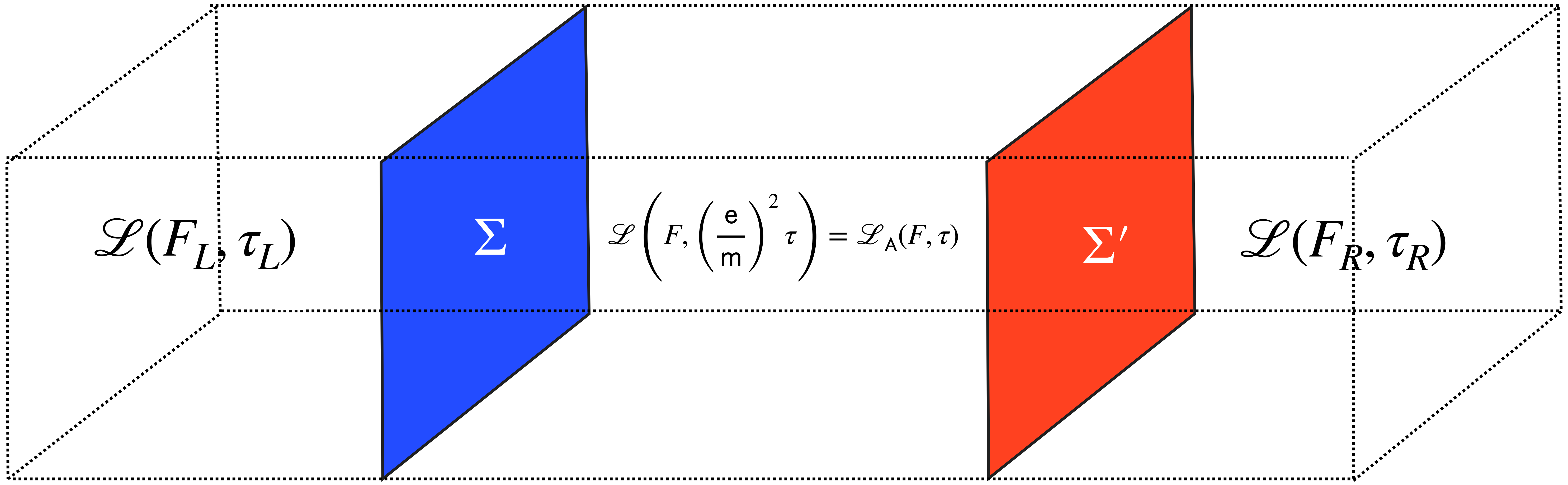

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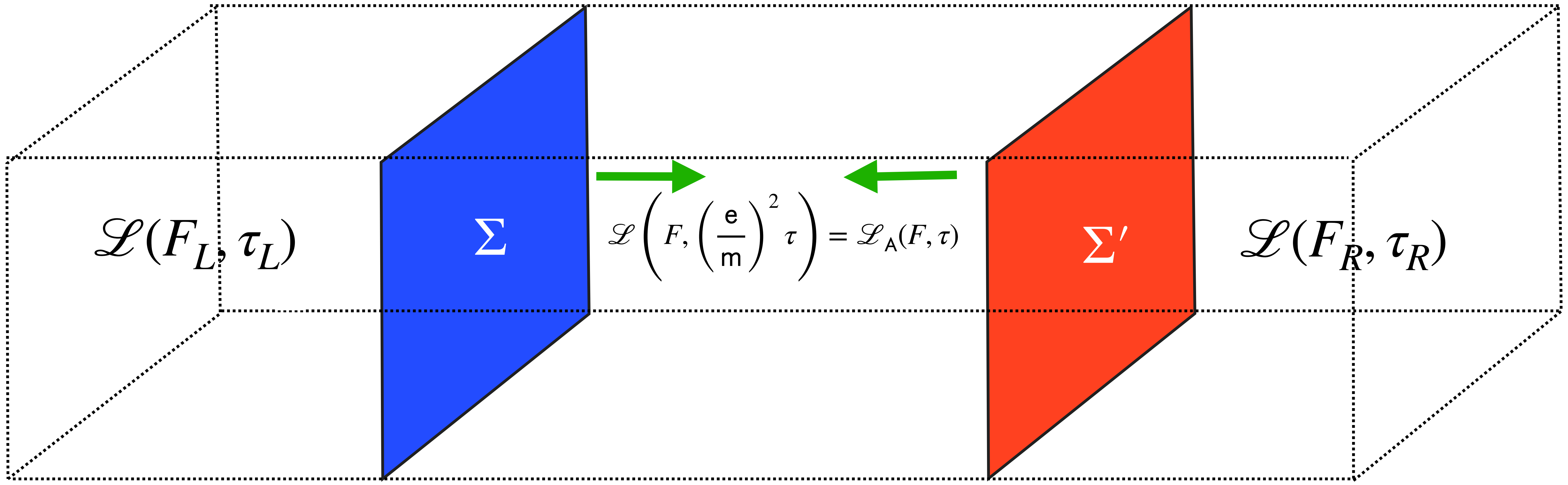

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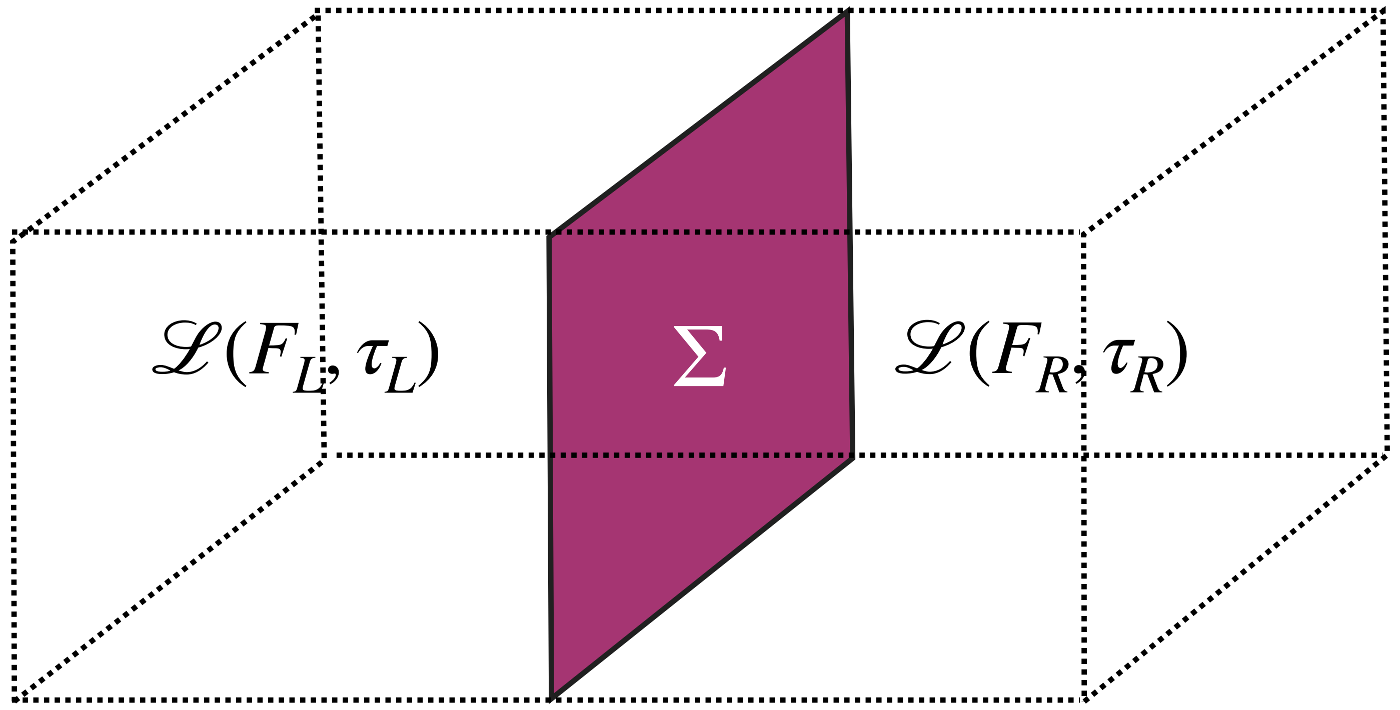

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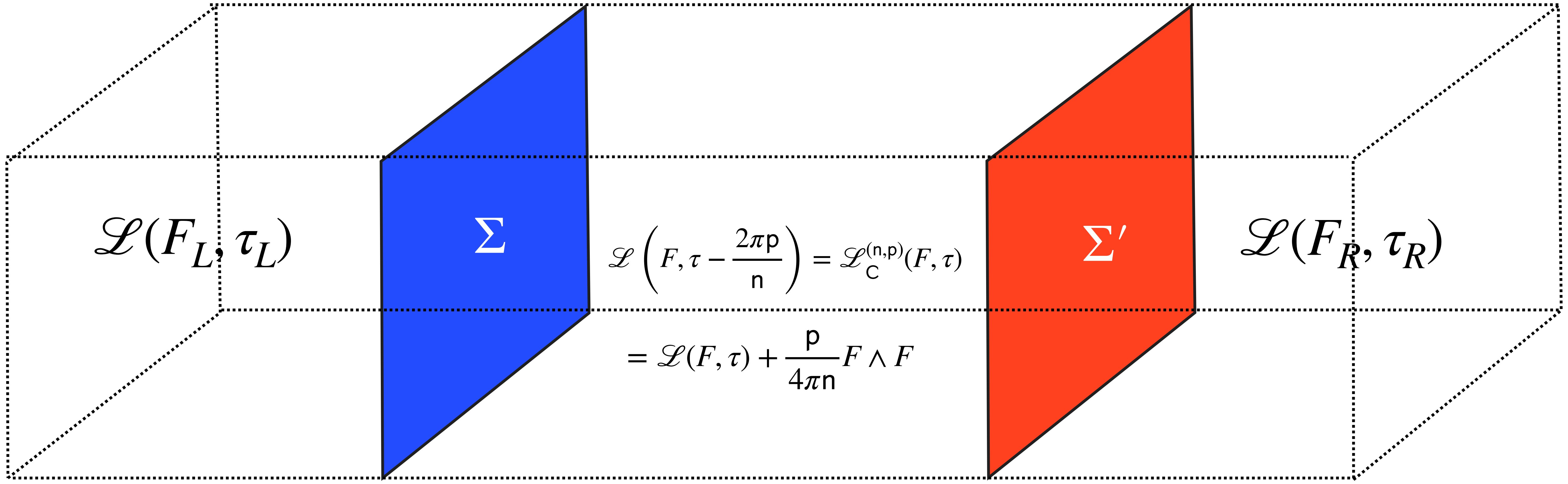
Gaillard-Zumino symmetry defects for Axion-Maxwell

Defect for $\mathscr{D}_{\mathsf{C}}$, where $\mathscr{A}_\Sigma^{(\mathsf{n},\mathsf{p})}\left[F_R/\mathsf{n}\right]$ is the minimal 3d TQFT coupled to F_R , $\left(\mathscr{A}_\Sigma^{(\mathsf{n},1)}\left[F_R/\mathsf{n}\right] = U(1)_n[F_R/\mathsf{n}]\right)$

[Choi, Lam, Shao '22]

$$\mathscr{U}_{\mathsf{C}}(\Sigma) = \exp\left[\frac{2i\mathsf{p}}{\mathsf{n}}\mathsf{f}^2\oint_\Sigma \frac{\star\, d\mathrm{Re}\tau}{(\mathrm{Im}\tau)^2}\right] \qquad \mathscr{W}_{\mathsf{C}}^{(\mathsf{n},\mathsf{p})}(\Sigma) = \mathscr{I}_\Sigma^{(\mathsf{n},\mathsf{p})}\left[F_R/\mathsf{n}\right]\int Db\ \exp\left[\frac{i}{2\pi}\oint_\Sigma b\wedge(F_R - F_L)\right]$$



$$(F_L, G_L, \vartheta_L)|_\Sigma = (F, G, \vartheta - 2\pi\mathsf{p}/\mathsf{n})|_\Sigma \qquad (F, G, \vartheta)|_\Sigma = (F_R, G_R + (\mathsf{p}/\mathsf{n})F_R, \vartheta_R)|_\Sigma$$

$\gcd(\mathsf{p},\mathsf{n}) = 1 \Rightarrow \mathscr{I}_\Sigma^{(\mathsf{n},\mathsf{p})}\left[F_R/\mathsf{n}\right] = \mathscr{A}_\Sigma^{(\mathsf{n},\mathsf{p})}\left[F_R/\mathsf{n}\right]$

$\gcd(\mathsf{p},\mathsf{n}) \neq 1 \Rightarrow \mathscr{I}_\Sigma^{(\mathsf{n},\mathsf{p})}\left[F_R/\mathsf{n}\right] \supset \exp\left(-\frac{i\mathsf{k}}{4\pi}\oint_\Sigma A_R\wedge F_R\right),$

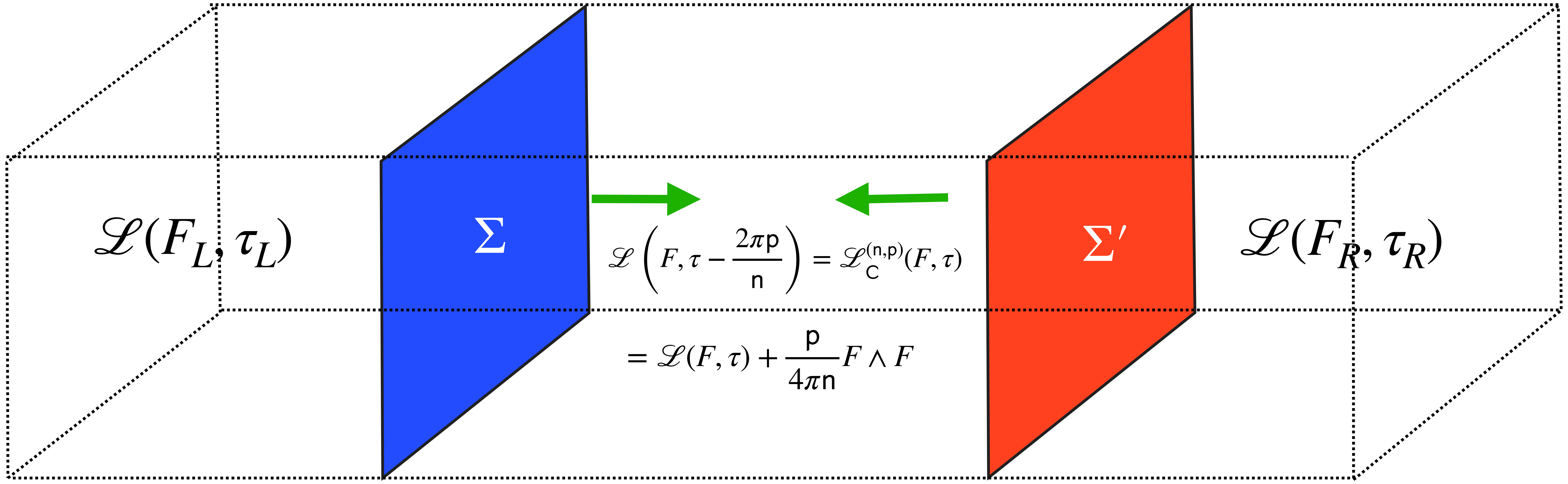
$\mathsf{k}\in\mathbb{Z}$

Gaillard-Zumino symmetry defects for Axion-Maxwell

Defect for $\mathscr{D}_{\mathsf{C}}$, where $\mathscr{A}^{(\mathsf{n},\mathsf{p})}_{\Sigma}\left[F_R/\mathsf{n}\right]$ is the minimal 3d TQFT coupled to F_R , $(\mathscr{A}^{(\mathsf{n},1)}_{\Sigma}\left[F_R/\mathsf{n}\right] = U(1)_n[F_R/\mathsf{n}])$

[Choi, Lam, Shao '22]

$$\mathscr{U}_{\mathsf{C}}(\Sigma) = \exp\left[\frac{2i\mathsf{p}}{\mathsf{n}}\mathsf{f}^2\oint_{\Sigma}\frac{\star\, d\mathrm{Re}\tau}{(\mathrm{Im}\tau)^2}\right] \qquad \mathscr{W}^{(\mathsf{n},\mathsf{p})}_{\mathsf{C}}(\Sigma) = \mathscr{I}^{(\mathsf{n},\mathsf{p})}_{\Sigma}\left[F_R/\mathsf{n}\right]\int Db\ \exp\left[\frac{i}{2\pi}\oint_{\Sigma} b\wedge(F_R - F_L)\right]$$



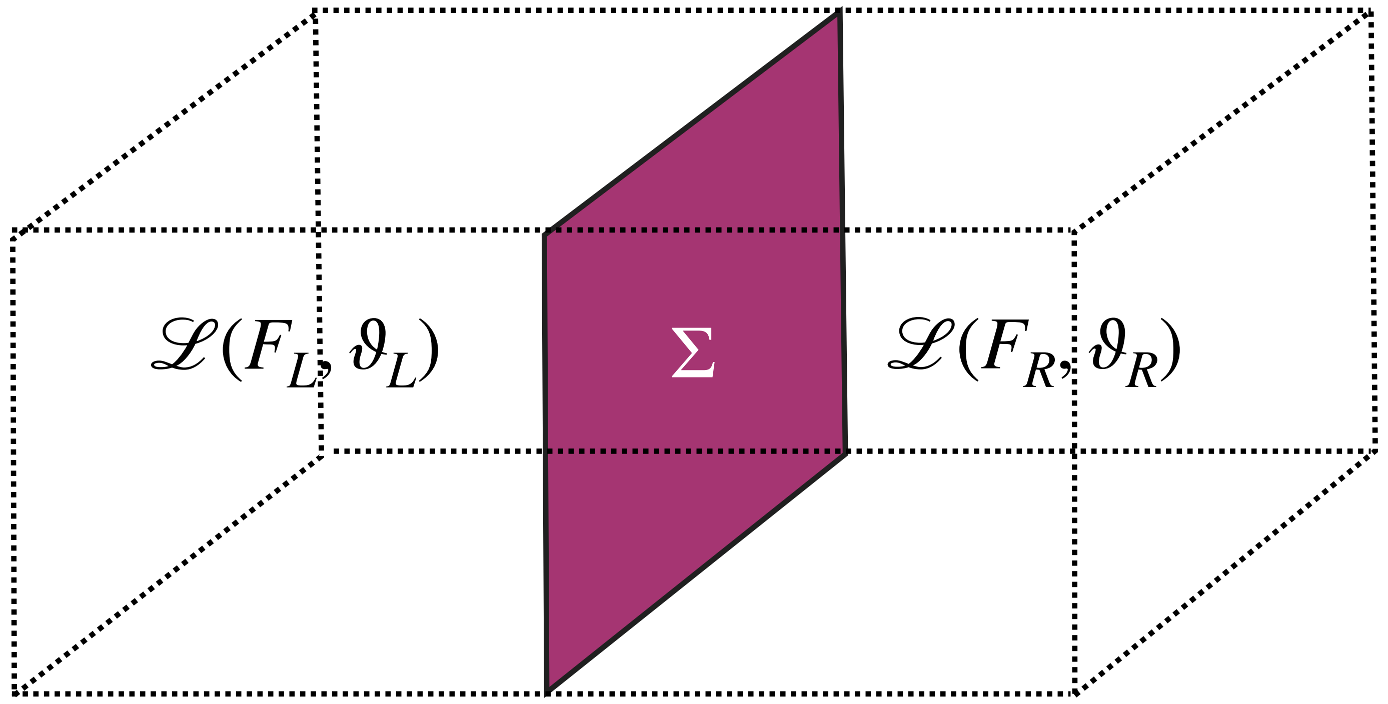
$(F_L, G_L, \vartheta_L)|_{\Sigma} = (F, G, \vartheta - 2\pi\mathsf{p}/\mathsf{n})|_{\Sigma}$ $\qquad$ $(F, G, \vartheta)|_{\Sigma} = (F_R, G_R + (\mathsf{p}/\mathsf{n})F_R, \vartheta_R)|_{\Sigma}$

$\gcd(\mathsf{p},\mathsf{n}) = 1 \Rightarrow \mathscr{I}^{(\mathsf{n},\mathsf{p})}_{\Sigma}\left[F_R/\mathsf{n}\right] = \mathscr{A}^{(\mathsf{n},\mathsf{p})}_{\Sigma}\left[F_R/\mathsf{n}\right]$

$\gcd(\mathsf{p},\mathsf{n}) \neq 1 \Rightarrow \mathscr{I}^{(\mathsf{n},\mathsf{p})}_{\Sigma}\left[F_R/\mathsf{n}\right] \supset \exp\left(-\frac{i\mathsf{k}}{4\pi}\oint_{\Sigma} A_R\wedge F_R\right),$

$\mathsf{k}\in\mathbb{Z}$

$$\mathscr{D}^{(\mathsf{n},\mathsf{p})}_{\mathsf{C}}(\Sigma) = \mathscr{U}^{(\mathsf{n},\mathsf{p})}_{\mathsf{C}}(\Sigma)\times\mathscr{W}^{(\mathsf{n},\mathsf{p})}_{\mathsf{C}}(\Sigma)$$



$(F_L, G_L, \vartheta_L)|_{\Sigma} = (F_R, G_R + (\mathsf{p}/\mathsf{n})F_R, \vartheta_R + 2\pi\mathsf{p}/\mathsf{n})|_{\Sigma}$