Strings and Geometry’26
Uppsala U., May 19-22, 2026

Symmetry Theories, Anomalies and η Invariants (Geometric Engineering and Quiver Approaches)

Mirjam Cvetič

Penn
University of Pennsylvania

CAMTP
University of Maribor

Univerza v Ljubljani
Fakulteta za matematiko in fiziko

Based on:

M.C., R. Donagi, J. Heckman, M. Hübner, 2512.17906 (geometric engineering)

&

V. Chakrabhavi, M.C., J. Heckman, S. Meynet, 2605….. (quivers)

[Also, earlier works (geometric engineering):

M.C., R. Donagi, J. Heckman, M. Hübner, E. Torres, 2408.12600

&

M.C., Heckman, Hübner, Torres, 2203.10102;

M.C., Heckman, Hübner, Torres, 2305.09665;

M.C., Heckman, Hübner, Torres, 2307.13027]

I. Motivation – Geometric Engineering Approach

Well-established study of Higher Symmetry Structures in Quantum Field Theory (QFT) via:

String/M theory on non-compact, singular spaces X

a) Typical approach: singular geometry X is resolved locally to smooth X' to great computational success

[Heckman, Park, Rudelius; 2015], [Morrison, Schafer-Nameki, Willett; 2020], [Bhardwaj, Schafer-Nameki; 2020], [Apruzzi, Bonetti, Garcıa-Etxebarria, Hosseini, Schafer-Nameki; 2021], [Tian, Wang; 2021], [M.C., Dierigl, Lin, Zhang; 2021], [Del Zotto, Garcıa-Etxebarria; 2022], [Lawrie, Yu, Zhang; 2023] & many more

b) However, key quantities often topological: calculations expected to be possible in singular spaces X via algebraic/differential topology

It may also be necessary to work with singular X due to the complexity of the singular geometry of X

Examples

- **Nested/non-Isolated Singularities**

  New results in singular geometry:
  [M.C., Heckman, Hübner, Torres; 2022], [Del Zotto,Garcıa-Etxebarria, Schafer-Nameki; 2022],
  [Acharya, Del Zotto, Heckman, Hübner, Torres; 2023],…,
  [M.C., Donagi, Heckman, Hübner, Torres; 2024], [M.C., Donagi, Heckman, Hübner; 2025]

- **Frozen Singularities**

  [de Boer, Dijkgraaf, Hori, Keurentjes, Morgan, Morrison, Sethi; 2001], [Atiyah, Witten; 2001], ...,
  New results in singular geometry:
  [M.C., Dierigl, Lin, Torres, Zhang; 2025]

- **Terminal Singularities**

  [Denef, Douglas, Florea, Grassi, Kachru; 2005], ...,
  New results in singular geometry:
  [Arras, Grassi, Weigand; 2016],...,
  [Sangiovanni, Valandro; 2024]

- **Non-SUSY String Constructions**

  [Adams, Polchinski, Silverstein; 2001], [Morrison, Narayan,Plesser; 2004], ...
  New results in singular geometry:
  [Braeger, Chakrabhavi, Heckman, Hübner; 2025]

Recent research focused on the analysis in singular X; contributions to the geometric engineering toolbox from differential cohomology, Chen-Ruan orbifold cohomology, equivariant cohomology, K-theory,...

II. Motivation – Quiver Approach

Less explored for studies of Higher Symmetry Structures

Anticipate that this data can be extracted from the path algebra of branes probing singular X.

Defect group [Del Zotto, Heckman, Meynet, Moscrop, Zhang; 2022]
[Del Zotto and Garcia-Etxebarria; 2022]…
[Dramburg, Meynet, Sangiovanni; 2025]

This approach provides a complementary algebraic approach to geometric engineering

Outline

I. Geometric Engineering Approach

i) Higher symmetry defects/operators

ii) Symmetry Field Theories (SymTFTs) and Nested Ones
(Isolated and non-isolated singularities examples)

iii) Anomaly terms in SymTFTs

iv) Computation of Anomaly Coefficients
(Isolated and non-isolated singularities examples)

Focus on refined data

Outline

II. Quiver Approach

- Present a procedure for extracting both the higher symmetries, as well as SymTFT anomaly coefficients (i-iv) directly from the singular space X, by focusing on the Quiver Data of quantum theory of branes, probing the singular geometry of X.

  [Focus on Quiver Data which directly tracks BPS defects in engineered QFT and its KK reduction.]

In the rest of the talk

M-theory on X – non-compact conical Calabi-Yau three-folds

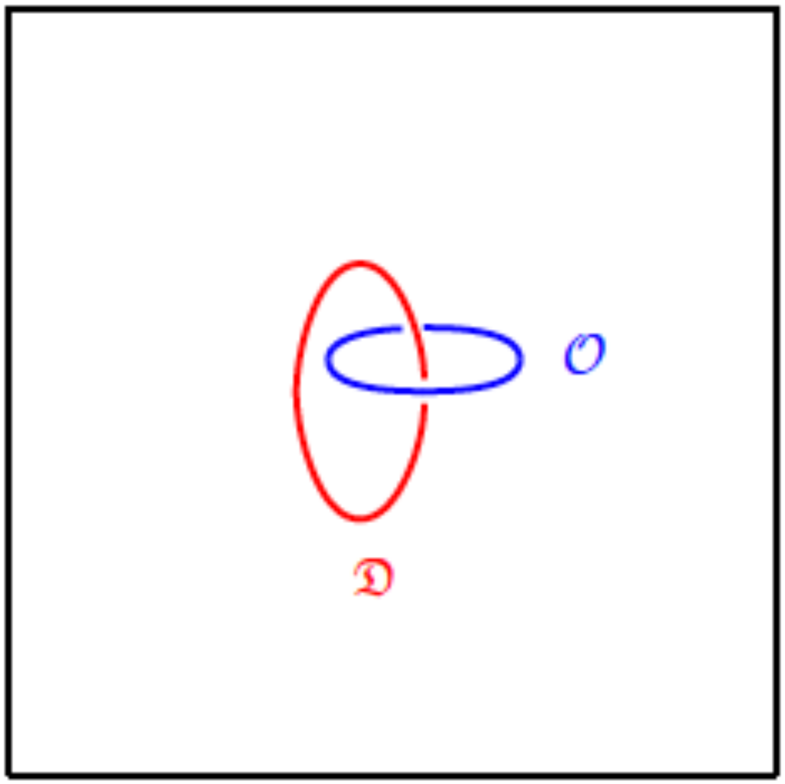
prototypes: orbifolds $X = \mathbb{C}^3/\Gamma$ with finite $\Gamma \subset SU(3)$

Supersymmetric QFT in 5D

[Gaiotto, Kapustin, Seiberg, Willet; 2014]

1) Higher Global Symmetries in QFT

Introduction of an extended defects $\mathfrak{D}$ and a symmetry operators $\mathcal{O}$ acting on them



Space-time

One-form symmetry generated by a string defect $\mathfrak{D}$ (1-dim) (Wilson line) and a symmetry operator $\mathcal{O}$ acting on it.

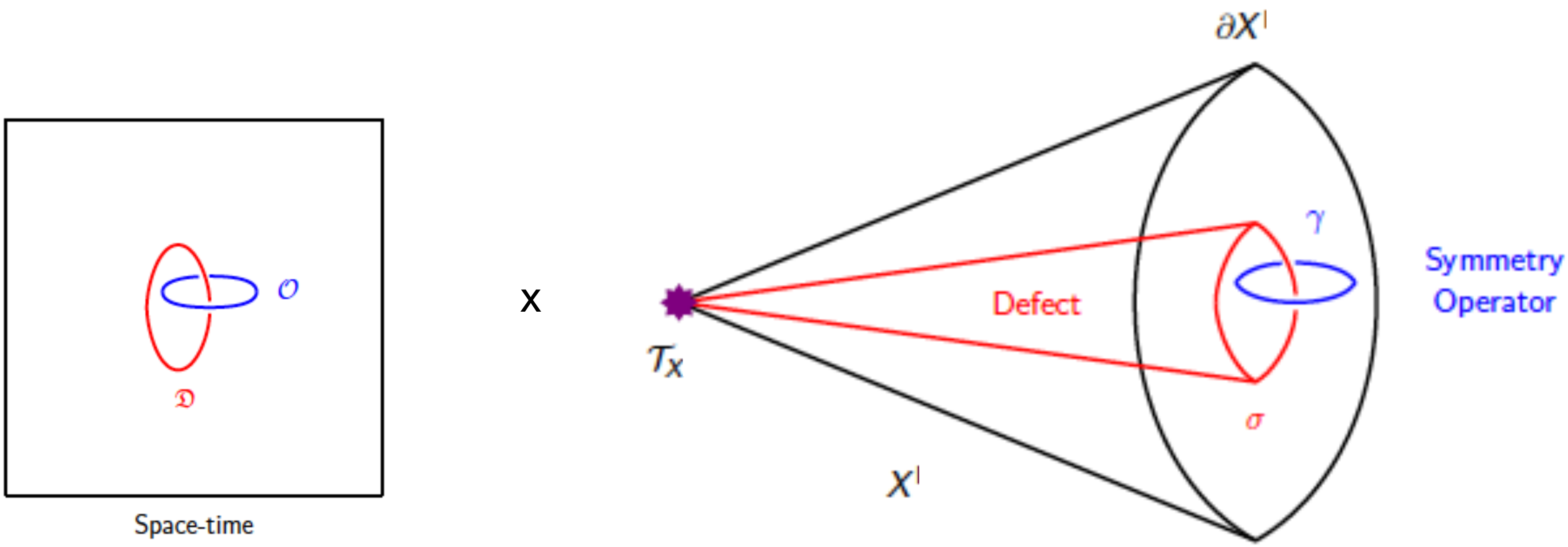
[Standard zero-form symmetry generated by a point defect $\mathfrak{D}$ (0-dim) & a symmetry operator $\mathcal{O}$ acting on it.]

Intertwining of one-form and zero-form symmetries → leads to two-group symmetry

[Benini, Cordova, Hsin; 2019],
[Lee, Ohmori, Tachikawa; 2021],…

Higher Symmetries from String Theory (Geometric Engineering)

Example: Isolated Singularity
QFT $\mathcal{T}_X$ with string theory construction on conical non-compact X



p-dim defects $\mathfrak{D}$ associated w/ branes wrapping non-compact cycles σ on X
Symmetry operators $\mathcal{O}$ w/ branes wrapping cycles γ on the boundary ∂X

Example: M-theory, defects due to M2 and M5 branes

El. $\mathcal{D}_p^{\text{M2}} = \dfrac{H_{3-p}(X,\partial X)}{H_{3-p}(X)} \cong H_{3-p-1}(\partial X)|_{\text{triv}}$

Mag. $\mathcal{D}_p^{\text{M5}} = \dfrac{H_{6-p}(X,\partial X)}{H_{6-p}(X)} \cong H_{6-p-1}(\partial X)|_{\text{triv}}$

Focus on electric (M2-branes) p=1 defects (Wilson lines)
Symmetry operators: $H_\ell(\partial X)$

[Del Zotto, Heckman, Park, Rudelius; '16]
[Morrison, Schafer-Nameki, Willett; '20]
[Albertini, Del Zotto, Etxebarria, Hosseini; '20]
…
[M.C., Heckman, Hübner, Torres; '22
[Del Zotto, Garcia Etxebarria, Schäfer-Nameki; 2022],
…
[Heckman, Hübner, Torres, Zhang; 2023],…
[M.C.,Heckman, Hübner, Torres; 2023],..

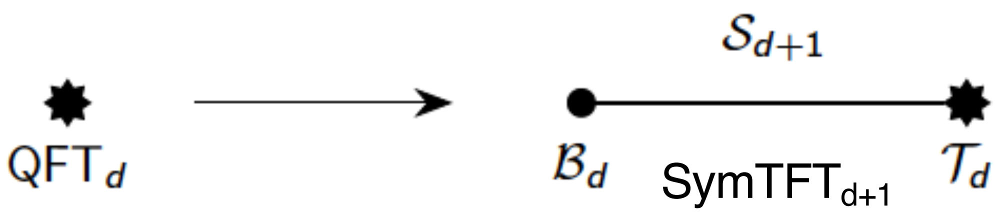
2a) Symmetry Theories

c.f., Ling Lin's talk…

A starting point to study how symmetries are represented (spontaneously broken or not?), including constraints on representations, anomalies, etc.

[Witten; 1998], [Gaiotto, Kulp; 2020];
[Apruzzi, Bonetti, Garcıa-Etxebarria, Hosseini, Schäfer-Nameki; 2021],
[Freed, Moore, Teleman; 2022], [Kaidi, Ohmori, Zheng; 2023];
[Brennan, Sun; 2024], [Bonetti, Del Zotto, Minasian; 2024], ...

In QFT in d-dim one can often isolate such symmetry structures into a topological field theory (TFT) in (d+1)-dim →
Symmetry Topological Field Theory (SymTFT) $\mathcal{S}_{d+1}$



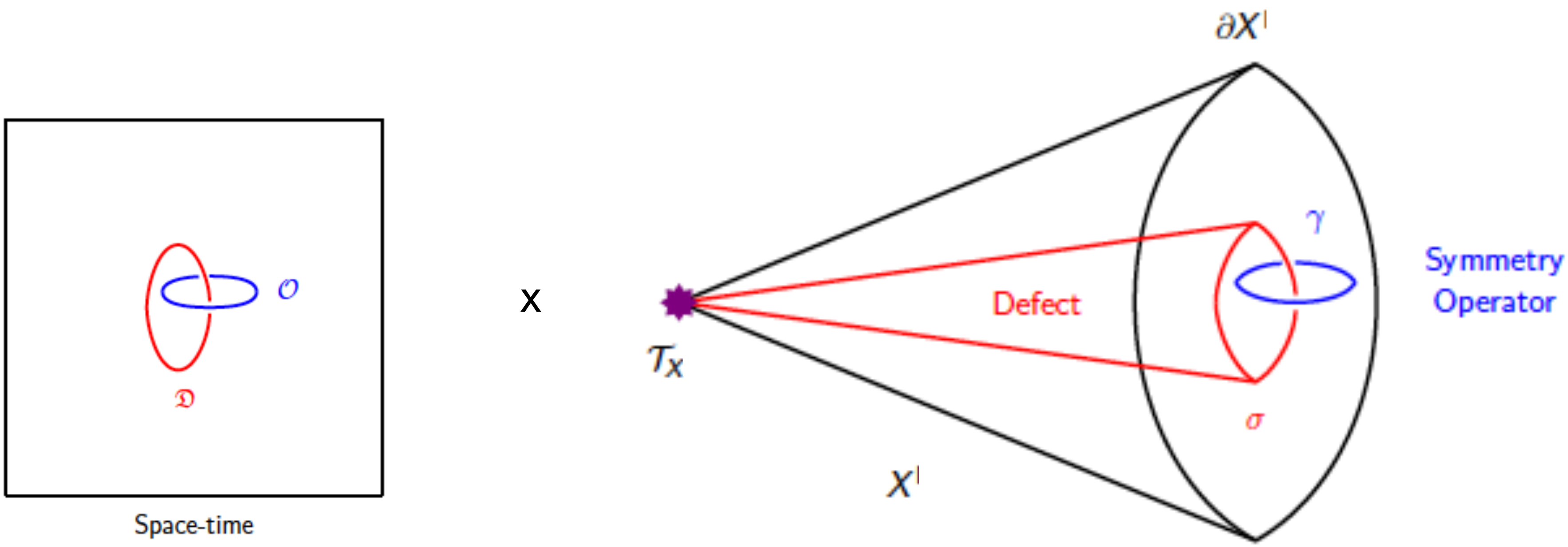
indicating physical boundary conditions (b.c.) at $\mathcal{T}$ and topological b.c. at $\mathcal{B}$ for the SymTFT.

Many symmetry structures depend only on $\mathcal{B}$ and the TFT.

SymTFT from String Theory (Geometric Engineering)

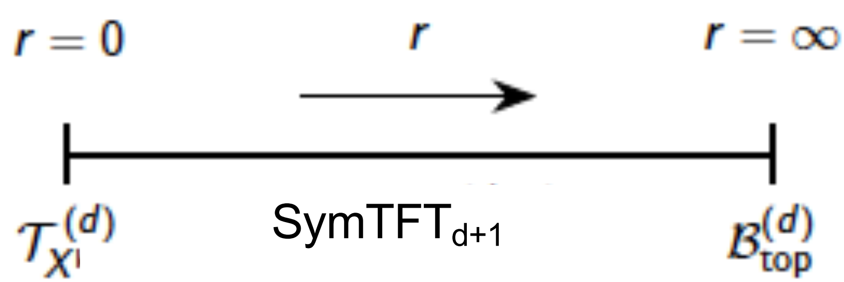
Prototype: Isolated Singularity

QFT $\mathcal{T}_X$ with string theory construction on conical non-compact X



Its SymTFT follows via compactification of ∂X over radial slices:

Physical b.c. from singularities,
topological b.c. from SUGRA b.c.



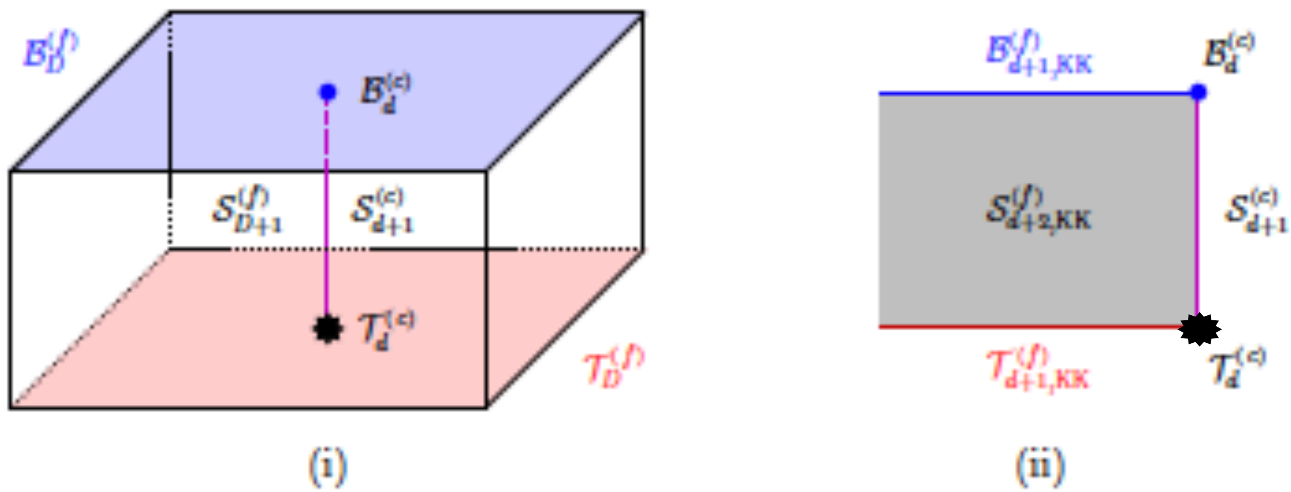
[Apruzzi, Bonetti, Garcia Etxebarria, Hosseini, Schäfer-Nameki; 2021]
[Heckman, Hübner, Torres, Yu, Zhang;2022][van Beest, Gould, Schäfer-Nameki, Wang; 20'22]
[Yu;2023] [Apruzzi, Bonetti, Gould, Schäfer-Nameki; 2023]
[Lawrie, Yu, Zhangl 2023] [Del Zotto, Meynet, Moscrop; 2024]...[Franco, Yu; 2024]…

2b) Generalizations to Nested SymTFTs

[M.C., Donagi, Heckman, Hübner, Torres; 2024]

Prototype

Consider a QFT in D dim and 'insert' into it a defect QFT in d<D dim.
What is the symmetry theory of the combined system?



The sandwich of the bulk QFT in D-dim with defect QFT in d-dim inserted. → KK reduce (i) to an open nested SymTFTs.

Nested SymTFTs via Geometric Engineering Natural

They are associated with nested/non-isolated singularities

What SymTFTs are to Isolated Singularities,

Nested SymTFTs are tonon-Isolated Singularities

[M.C., Donagi, Heckman, Hübner, Torres; 2024]

Subsequent work

[Garcia-Etxebarria, Huertes, Uranga; 2024],
[Huertes, Uranga; 2024].
[Bhardway, Copetti, Pajer, Schäfer-Nameki; 2024],
…

Conical Prototypes of non-Isolated Singularities

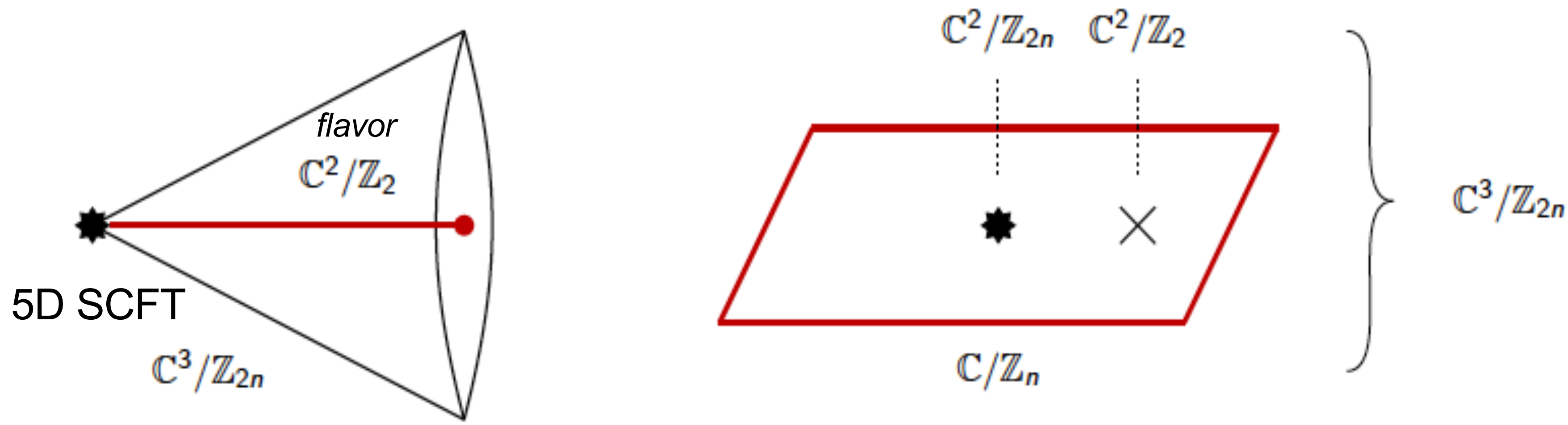
Orbifolds $\mathbb{C}^3/\Gamma$ = Cone(S^5/Γ) w/ finite $\Gamma \subset SU(3)$
w/flavor branes at co-dim 4 loci $\mathbb{C}^2/\Gamma_{fix}$= Cone($S^3/\Gamma_{fix}$)

$\mathbb{C}^3 = \{z_1, z_2, z_3\}$

$\Gamma = \mathbb{Z}_{2n}$, $\Gamma_{fix} = \mathbb{Z}_2$

Example:

- Setup: M-theory on $X = \mathbb{C}^3/\mathbb{Z}_{2n}(1,1,2n-2)$
- Two equivalent presentations of the geometry:



(i) Cone (ii) Fibration over z_3

- SU(n) 5D SCFT at $\{z_1, z_2, z_3\} = 0$ → isolated co-dim 6 singularity
  SU(2) flavor brane at $\{z_1, z_2\} = 0$ → red cone - co-dim 4 singularity
  (flavor brane $\mathbb{C}^2/\mathbb{Z}_2$)
- SU(n) 5D SCFT as defect nested within a 7D SU(2) SYM theory

Past

[M.C. Heckman, Hübner, Torres; 22]

Such non-compact orbifold examples were chosen to identify geometric origin of higher-form symmetries (0-form, 1-form & 2-group) by studying the symmetry defects via algebraic topology (via cutting & gluing of singular boundary ∂X of the non-compact space X *and* employing Mayer-Vietoris exact sequences).

Subsequent

[M.C. Heckman, Hübner, Torres; 24]

Concrete construction of the nested SymTFTs of the joint 7D/5D system. It reproduced old results, but the framework is now more general as one can study how any topological operator behaves when pushed from the symmetry bulk SymTFT onto the symmetry boundary SymTFT.

Further works [Garcia-Etxebarria, Huertes, Uranga; 2024], [Huertes, Uranga; 2024], [Bhardway, Copetti, Pajer, Schäfer-Nameki; 2024],…

Recent

[M.C., Donagi, Heckman, Hübner, Torres; 2024], [M.C., Donagi, Heckman, Hübner; 2025]

3) Anomaly Couplings in SymTFTs

Recall: M-theory on $R^{1,d}$ *x* X, with X = Cone(∂X) d+1=5, dim(X) = 6

The SymTFT on $R^{1,d}$ *x I* follows by compactification on ∂X.

Starting point for anomaly couplings: 11D SG topological terms

[Witten; 2001],…

$$\frac{2\pi}{6}\int_{11D}\frac{C_3}{2\pi}\left[\left(\frac{G_4}{2\pi}\right)^2+\frac{p_1^2-4p_2}{32}\right] \Rightarrow \int_{12D} L_{CS}=\frac{2\pi}{6}\int_{12D}\frac{G_4}{2\pi}\left[\left(\frac{G_4}{2\pi}\right)^2+\frac{p_1^2-4p_2}{32}\right]$$

Globally defined

Chern-Simons action in 12D

$$\Rightarrow \quad S_{\text{anomaly}}=\int_{\mathbb{R}^{1,d}\times I}\left(\int_{X'} L_{CS}\right)$$

SymTFT action

w/ ∂X′ = ∂X, but possibly X≠ X′ (topological coupling)

G_4- field strength of C_3; $p_{1,2}$ - Pontryagin classes of TM_{11}, specifying X_8 -8-form characteristic class

Anomaly couplings: Isolated Singularity

Recall: M-theory on $X = \mathbb{C}^3/\Gamma$ with finite $\Gamma \subset SU(3)$.
Symmetries include 1-form symmetry with background fields B_2, by expanding

$G_4 = G^0_4 + B_2F_2 + \ldots$

w/ free class $F_2 \in H^2(X')=\mathbb{Z}^\#$ projecting to a generator $H^2(X')/H^2(X',\partial X')\simeq \mathrm{Ab}(\Gamma)$.

[Apruzzi, Bonetti, Garcıa-Etxebarria, Hosseini, Schäfer-Nameki; 2021]

There is 1-form self-anomaly & mixed 1-form gravitational anomaly

$$S_{\text{anomaly}} = \int \frac{|\Gamma|}{2\pi} B_2 dB_3 + \boxed{\beta B_2^3 + \gamma B_2 (p_1/4)} + \ldots$$

$$\beta = \int_{X'} F_2^3$$

$$\gamma = \int_{X'} -\frac{1}{24} F_2 p_1(TX')$$

Calculation of Anomaly Coefficients β and γ
(determined up to shifts that change S_{anomaly} by an integer)

For BF-type terms see [Garcia-Etxebarria, Hosseini; 2024]

Calculation of Coefficient $\alpha = \beta + \gamma$

Employ Dirac operator on X′ coupled to a line bundle L with first Chern class $c_1(L) = F_2$. Its index, due to Atiyah-Patodi-Singer (APS):

[Atiyah, Patodi, Singer;1975]

bulk - denote $\alpha = \beta + \gamma$ boundary

$$\text{Index } D_L = \int_{X'} \left(\frac{1}{6} c_1(L)^3 - \frac{1}{24} p_1(TX') c_1(L) \right) - \frac{h_L(\partial X') + \eta_L^{\not D}(\partial X')}{2}$$

$$= \alpha - \frac{1}{2} \eta_L^{\not D}(\partial X')$$

$\eta_L^{\not D}(\partial X')$ - spectral invariant of Dirac op. (twisted by L) on ∂X′

[$h_L(\partial X') = 0$ – dim of kernel of Dirac op. on ∂X′]

- Integer & ∂ X′ = ∂X

Bulk coeff. $\alpha = \beta + \gamma$ (modulo integers) is

$$\alpha = \frac{1}{2} \eta_L^{\not D}(\partial X)$$

For $\mathbb{C}^3/\Gamma$ w/$\Gamma = Z_N(p, q, r)$ w/isolated singularities & $\partial X = S^5/\mathbb{Z}_N$:

$$\alpha = \frac{1}{N} \sum_{k=1}^{N-1} \frac{(-1)^k \omega^k}{(\omega^{-pk/2} - \omega^{pk/2})(\omega^{-kq/2} - \omega^{kq/2})(\omega^{-kr/2} - \omega^{kr/2})}$$

w/ $\omega = \exp(2\pi i/N)$ & L w/ Q=1

[Donnelly; 1978], [Gilkey; 1984], [Bär; 1996]

The Status of α Coefficient Calculations

Old: Resolution of X → X' & intersection theory (in the bulk)

[Jefferson, Katz, Kim, Vafa; 2018], [Bhardwaj, Jefferson; 2019], [Tian, Wang; 2021]…

→ applied to the calculation of α

[Apruzzi, Bonetti, Garcia Etxebarria, Hosseini, Schäfer-Nameki; 2021]

[M.C., Donagi, Heckman, Hübner; 2025]

Agree

New: Calculation of $\eta_L^{\not D}(\partial X)$ invariant for singular X (on the boundary)

& APS index $\alpha = \frac{1}{2}\eta_L^{\not D}(\partial X)$ [M.C., Donagi, Heckman, Hübner; 2025]

Concrete example: $\mathbb{C}^3/\mathbb{Z}_{2n+1}$ (1, 1, 2n–1):

$$\alpha = \frac{1}{2}\eta_L^{\not D}(S^5/\mathbb{Z}_{2n+1}) = \frac{n(n+1)}{6(2n+1)}$$

Results match for all $\mathbb{C}^3/\Gamma$, w/$\Gamma = \mathbb{Z}_N(p, q, r)$ & isolated singularities

Anomaly Coefficients β & γ

Anomaly coefficients β & γ depend on a choice of generator for Z_N.
For another generator, they are changed by rescaling:

$$B_2 \rightarrow Q\, B_2 ,$$

for any charge Q with gcd(N,Q) = 1.
Then he anomaly coefficients β & γ scale as

$$(\beta , \gamma) \rightarrow (Q^3 \beta, Q \gamma) ,$$

Thus: $\alpha_Q = Q^3 \beta + Q \gamma$ w/

$$\alpha_Q = \frac{1}{2}\eta^{\not{D}}_{L_Q} = \frac{1}{N}\sum_{k=1}^{N-1}(-1)^k \omega^{kQ} \prod_{i=1}^{3} \frac{1}{\omega^{-km_i/2} - \omega^{km_i/2}} \quad \text{mod } 1$$

$L_Q = (R_Q \times S^5)/Z_N$ for the charge Q representation R_Q of Z_N

[Q = q is a preferred generator → determined à la Kawasaki (no time)]

Can solve for β & γ :

$$\beta_{L_Q} = +\frac{1}{12}\eta^{\not{D}}_{L_{2Q}}(\partial X) - \frac{1}{6}\eta^{\not{D}}_{L_Q}(\partial X) \qquad \text{mod } 1$$

$$\gamma_{L_Q} = -\frac{1}{12}\eta^{\not{D}}_{L_{2Q}}(\partial X) + \frac{2}{3}\eta^{\not{D}}_{L_Q}(\partial X) \qquad \text{mod } 1$$

Anomaly Coefficients α, β & γ

One parameter Examples of Anomaly Coefficients for $C^3/Z_N(m_1,m_2,m_3)$, twisted by a line bundle L_q , w/ Q= q – Kawasaki preferred generator.

$\mathbb{Z}_N(m_1,m_2,m_3)$	$\frac{1}{2}\eta^{\not D}_{L_q}(S^5/\mathbb{Z}_N)$	q	β_{L_q}	γ_{L_q}
$\frac{1}{2n+3}(1,1,2n+1)$	$\frac{(n+1)(n-1)}{3(2n+3)}$	-2	$-\frac{2n-1}{6(2n+3)}$	$\frac{2n^2+2n-3}{6(2n+3)}$
$\frac{1}{12n+5}(1,3,12n+1)$	$\frac{(2n-1)(6n-10)}{12n+5}$	-12	$\frac{24}{12n+5}$	$\frac{12n^2-26n-14}{12n+5}$
$\frac{1}{30n+7}(1,5,30n+1)$	$\frac{(5n-9)(15n-11)}{30n+7}$	-30	$-\frac{5(30n-53)}{2(30n+7)}$	$\frac{150n^2-230n-67}{2(30n+7)}$
$\frac{1}{56n+9}(1,7,56n+1)$	$\frac{4(7n-13)(28n-23)}{3(56n+9)}$	-56	$-\frac{14(56n-103)}{3(56n+9)}$	$\frac{2(392n^2-658n-123)}{3(56n+9)}$

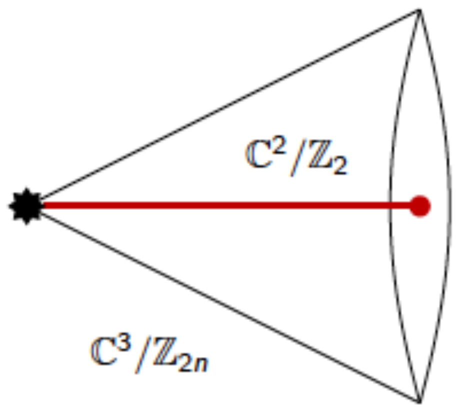
Explicit form also for a broader class of two parameter examples $C^3/Z_N(m_1,m_2,m_3)$, w/ isolated singularities → in the paper

All in agreement w/resolved calculations

Coefficients α for Non-Isolated Singularities

Recall: conical orbifolds $\mathbb{C}^3/\Gamma$ = Cone(S^5/Γ) w/ finite $\Gamma \subset SU(3)$
w/flavor branes at co-dim 4 loci $\mathbb{C}^2/\Gamma_{fix}$ = Cone(S^3/Γ_{fix})

Again, Concrete Example $\mathbb{Z}_{2n}$(1,1, 2n-2) orbifold: $\Gamma = \mathbb{Z}_{2n}$, $\Gamma_{fix} = \mathbb{Z}_2$



- SU(n) 5D SCFT at ✸ isolated co-dim 6 singularity
- SU(2) 7D SYM at red cone co-dim 4 singularity (flavor brane at $\mathbb{C}^2/\mathbb{Z}_2$)

Coefficients α for Non-Isolated Singularities

Example $\mathbb{Z}_{2n}(1,1,2n-2)$ orbifold:

$\Gamma = \mathbb{Z}_{2n}$, $\Gamma_{fix} = \mathbb{Z}_2$

- Calculation directly in field theory

[Gukov, Hsin, Pei; 2020]

Agree

- Resolution and intersection computation

[Apruzzi, Bonetti, Garcia Etxebarria, Hosseini, Schäfer-Nameki; 2021]

$$\alpha = \frac{(n-1)(n-2)}{6n}$$

Agree

- Computation directly in singular geometry:

[M.C. Donagi, Heckman, Hübner; 2025]

$$\alpha = \frac{1}{2}\eta_L^{D\!\!\!/}(S^5/\Gamma)$$

$$\frac{1}{2}\eta_L^{D\!\!\!/}(S^5/\Gamma) = \frac{1}{2n}\sum_{k=1,k\neq n}^{2n-1}\frac{(-1)^k\omega^{k(-2)}}{(\omega^{-k/2}-\omega^{k/2})(\omega^{-k/2}-\omega^{k/2})(\omega^{-k(2n-2)/2}-\omega^{k(2n-2)/2})}$$

[Degaratu; 2008]

$\omega = \exp(2\pi i/N)$ & L_Q w/ $Q = -2$

Anomaly Coefficients : β & γ for $\Gamma = \mathbb{Z}_N$, $\Gamma_{fix} = \mathbb{Z}_g$

Eta Invariance for & L_Q

[Degaratu; 2008]

$$\frac{1}{2}\eta_L^{D\!\!\!/} = \frac{1}{N}\sum_{\gamma\in \mathrm{FF}(\mathbb{Z}_N)} (-1)^k \omega^{kQ_L} \prod_{i=1}^{3} \frac{1}{\omega^{-km_i/2} - \omega^{km_i/2}}$$

$$= \frac{1}{N}\sum_{\ell=0}^{g-1}\left(\sum_{k=\ell N/g+1}^{(\ell+1)N/g-1} (-1)^k \omega^{kQ_L} \prod_{i=1}^{3} \frac{1}{\omega^{-km_i/2} - \omega^{km_i/2}}\right)$$

$$\alpha_L = \frac{1}{2}\eta_L^{D\!\!\!/} \mod 1$$

$$\alpha_L = \beta_L + \gamma_L \qquad \beta_{L_Q} = +\frac{1}{12}\eta^{D\!\!\!/}_{L_{2Q}}(\partial X) - \frac{1}{6}\eta^{D\!\!\!/}_{L_Q}(\partial X) \mod 1$$

$$\gamma_{L_Q} = -\frac{1}{12}\eta^{D\!\!\!/}_{L_{2Q}}(\partial X) + \frac{2}{3}\eta^{D\!\!\!/}_{L_Q}(\partial X) \mod 1$$

One-parameter examples:

$\mathbb{Z}_N(m_1, m_2, m_3)$	$\frac{1}{2}\eta_{\dot{L}_q}'(S^5/\mathbb{Z}_N)$	q	β_{L_q}	γ_{L_q}
$\frac{1}{2n+2}(1,1,2n)$	$\frac{n(n-1)}{6(n+1)}$	-2	$-\frac{n-1}{6(n+1)}$	$\frac{n^2-1}{6(n+1)}$
$\frac{1}{6n+3}(1,2,6n)$	$\frac{n(n-1)}{2(2n+1)}$	-3	$-\frac{n-1}{6(2n+1)}$	$\frac{3n^2-2n-1}{6(2n+1)}$
$\frac{1}{12n+4}(1,3,12n)$	$\frac{n(n-1)}{3n+1}$	-4	$-\frac{n-1}{6(3n+1)}$	$\frac{6n^2-5n-1}{6(3n+1)}$
$\frac{1}{20n+5}(1,4,20n)$	$\frac{5n(n-1)}{3(4n+1)}$	-5	$-\frac{n-1}{6(4n+1)}$	$\frac{10n^2-9n-1}{6(4n+1)}$

In agreement w/ explicit intersection calculations for resolved geom.

Also examples with non-cyclic/non-Abelian Γ & Z_N x Z_M → in the paper

For non-isolated (flavor) singularities

Non-Abelian flavor symmetry for $\mathbb{C}^3/\mathbb{Z}_N(m_1, m_2, m_3)$ w/ $g_k \equiv \gcd(N, m_k) > 1$

$$\rightarrow \quad \mathfrak{f} \equiv \mathfrak{su}(g_1) \oplus \mathfrak{su}(g_2) \oplus \mathfrak{su}(g_3)$$

Additional (Mixed) Flavor Brane Anomalies

[Via Wess-Zumino Action/7D SYM effective action reduction]

$$\frac{2\pi}{2(2\pi)^2}\sum_{k=1}^{3}\delta_k\int_{6\mathrm{D}} B_2^{(N/g)}\mathrm{Tr}(F_2^{(k)}F_2^{(k)}) \quad \text{and} \quad \frac{2\pi}{6(2\pi)^3}\sum_{k=1}^{3}\epsilon_k\int_{6\mathrm{D}}\mathrm{Tr}(F_2^{(k)}F_2^{(k)}F_2^{(k)})$$

$$\delta_k = \ell(t_4^{(ij)}, t_2) = q g_i g_j / N\,, \qquad \{i,j,k\} = \{1,2,3\} \qquad\qquad \epsilon_k = \frac{m_k}{N}$$

[M.C., R. Donagi, J. Heckman, M. Hübner, E. Torres; 2024]

II. Quiver Approach – Focus on Anomaly Coefficients

[V. Chakrabhavi, M.C., J. Heckman, S. Meynet; 2605…..]

Concretely: Study 4D Type IIA (by reducing M-theory of X on S^1)
w/ D0 probing X, realizing a quiver quantum mechanics or
T^3-dual to Type IIB w/ D3 probing X, realizing D=4, N=1 quiver QFT

For $X = \mathbb{C}^n/\Gamma$ with boundary $\partial X = S^{2n-1}/\Gamma$: (n=3)

$$\frac{1}{2}\eta_\chi^{\bar\partial}\left(S^{2n-1}/\Gamma\right) = \frac{(-1)^n}{|\Gamma|}\sum_{\gamma\in\Gamma_{\mathbf{FF}}}\frac{\chi(\gamma)}{\det(1-\gamma)}$$

[Degaratu; 2008]

w/ canonical lift $\Gamma \to SU(n)$, representation of Γ w/ character χ &

$\Gamma_{\mathbf{FF}} \subset \Gamma$ the subset acting fixed-point freely on the sphere S^{2n-1}

Degaratu's Theorem:

$$\frac{1}{2}\eta_\chi^{\bar\partial}(S^{2n-1}/\Gamma) = (-1)^n \operatorname*{Res}_{t=1}\frac{M_\chi(t)}{1-t}$$

w/ $M_\chi(t)$ denotes the twisted (by character χ) Molien series:

$$M_\chi(t) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\frac{\chi(\gamma)}{\det(1-t\gamma)}$$

Untwisted Molien series ($\chi = 1$) – generating function,
counting the number of Γ- invariant holomorphic functions
(independent polynomial invariants at each polynomial degree) for $X = \mathbb{C}^n/\Gamma$.

Twisted Molien series – generating function, counting symmetric polynomials of specific degrees, "twisted" by a finite group action.

Note, anomaly takes the form:

$$\alpha = \frac{1}{2}\eta_L^{\not D}(Y) = \frac{1}{2}\left(\eta_L^{\bar\partial}(Y) - \eta_\varnothing^{\bar\partial}(Y)\right)$$

$Y = \partial X = S^{2n-1}/\Gamma$
(n=3)

w/ $\eta_L^{\bar\partial}$ & $\eta_\varnothing^{\bar\partial}$ expressed in terms twisted and untwisted Molien series as:

$$\alpha = \underset{t=1}{\mathrm{Res}} \frac{M_{\mathrm{tw}}(t) - M_{\mathrm{untw}}(t)}{1-t}$$

From Geometric Engineering to Representation Theory!

Example: X= $\mathbb{C}^3/\Gamma$ w/ $\Gamma=\mathbb{Z}_3$:

$$M_{\mathrm{tw}}(t) = \frac{1}{3}\sum_{k=0}^{2} \frac{\omega^{-k}}{(1-\omega^k t)^3} = \frac{3(t+2t^4)}{(1-t^3)^3}$$

$$M_{\mathrm{untw}}(t) = \frac{1}{3}\sum_{k=0}^{2} \frac{1}{(1-\omega^k t)^3} = \frac{1+7t^3+t^6}{(1-t^3)^3}$$

$\omega = e^{2\pi i/3}$.

$\alpha = -1/9$

agrees w/geometric engineering

Quiver Hilbert Series and η Invariants

- Introduce matrix-valued Hilbert series H(t), derived from the brane-probe of X (quiver-theoretic refinement of the Molien series).

[Eager; 2011],
also [Ginzburg; 2006], [Bocklandt;2006],[Martelli, Sparks, Yau;2006]

- w/ matrix H(t)'s (i,j)- entry counts (F-term equivalence) classes of paths from quiver node i to node j (weighted by total R-charge).

Diagonal entries(i=j) count loops at a given quiver node i;
off-diagonal ones (i≠j) count open paths between i and j, interpolating between distinct fractional branes.

- For orbifolds $\mathbb{C}^3/\Gamma$, w/{R_i} – irreducible representations of Γ,
  R – representation by which Γ acts on $\mathbb{C}^3$
  A_{ij} - the McKay adjacency matrix defined as

[McKay;1980],
[Kronheimer, Nakajima;1990]…
[Douglas, Moore; 1996],
[Lawrence, Nekrasov, Vafa;1998]..

$$R \otimes R_i = \bigoplus_j A_{ij} \; R_j$$

matrix-valued Hilbert series greatly simplifies:

$$H(t) = \left(\mathbf{1} - tA + t^2 A^T - t^3 \mathbf{1}\right)^{-1}$$

Connections to Molien series

$H_{00}(t)$ (trivial rep. node) correspond to untwisted Molien series:

$$H_{00}(t) = \frac{1}{|\Gamma|}\sum_{g\in\Gamma}\frac{1}{\det(1-t\rho(g))}$$

ρ(g) =γ from before

$H_{ij}(t)$ correspond to twisted Molien series:

$$H_{ij}(t) = \frac{1}{|\Gamma|}\sum_{g\in\Gamma}\frac{\chi_i(g)\chi_j(g^{-1})}{\det(1-t\rho(g))}$$

w/ χ_i (g) is the character of the irreducible representation R_i.

$$M_{\rm untw}(t) = H_{00}(t), \qquad M_{\rm tw}(t) = H_{0i}(t) \qquad (i\neq 0)$$

Anomaly coefficient from quiver data

$$\alpha_i = \operatorname*{Res}_{t=1}\frac{H_{0i}(t)-H_{00}(t)}{1-t} \quad \bmod 1,$$

[i =1 for isolated sing]

$H_{00}(t)$ counts gauge-invariant operators,
= H_{untw}(t)

$H_{0i}(t)$ counts operators in the ith twisted sector
= H_{tw}(t)

Quiver probing $\mathbb{C}^3/\mathbb{Z}_3$

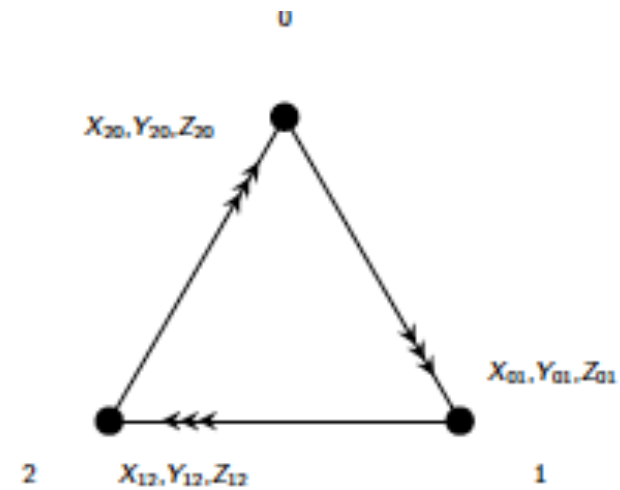


Figure: Quiver for D-brane probe of $\mathbb{C}^3/\mathbb{Z}_3$.

Adjacency matrix A:

$$A = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 3 & 0 & 0 \end{pmatrix}$$

Matrix valued Hilbert series H(t):

$$H(t) = \begin{pmatrix} -\frac{1+7t^3+t^6}{(-1+t^3)^3} & -\frac{3(t+2t^4)}{(-1+t^3)^3} & -\frac{3t^2(2+t^3)}{(-1+t^3)^3} \\ -\frac{3t^2(2+t^3)}{(-1+t^3)^3} & -\frac{1+7t^3+t^6}{(-1+t^3)^3} & -\frac{3(t+2t^4)}{(-1+t^3)^3} \\ -\frac{3(t+2t^4)}{(-1+t^3)^3} & -\frac{3t^2(2+t^3)}{(-1+t^3)^3} & -\frac{1+7t^3+t^6}{(-1+t^3)^3} \end{pmatrix}$$

$$\alpha = \operatorname*{Res}_{t=1} \frac{H_{01}(t) - H_{00}(t)}{1-t} = -1/9$$

Agrees with geometric engineering

Final Remarks

- The η-invariant anomaly coefficient extracted from a matrix-valued quiver Hilbert series H(t) also for non-isolated (flavor singularities); α for all Abelian orbifolds via H(t).

- Approach valid for quivers probing other toric CYs,
  e.g., cones over $Y^{p,q}$, [Martelli, Gauntlett, Sparks, Waldram;2004]
  $L^{p,q,r}$ [M.C., Lü, Page, Pope; 2005] ,…
  → new results for η invariants for families of cones over $Y^{p,q}$

- For non-isolated singularities, additional flavor sectors are supported on lower-dimensional strata. (These flavor symmetries can mix with the one-form symmetry through a 2-group structure.) This refined data is captured by tracking the higher-order poles:

$$\alpha^{\text{ref}} = \frac{1}{2}\eta^{\text{ref}} = \operatorname{Res}^{(2)}_{t=1}\left[\frac{H_{\text{tw}}(t) - H_{\text{untw}}(t)}{1-t}\right] + \operatorname*{Res}_{t=1}\left[\frac{H_{\text{tw}}(t) - H_{\text{untw}}(t)}{1-t}\right] \mod 1$$

$\text{Res}^{(2)}$ captures data from the $(1 - t)^{-2}$ term

w/ $$\alpha^{\text{ref}} \equiv \beta + \gamma + \sum_k \delta_k + \sum_k \epsilon_k \mod 1$$

Thank you!