Anomalies and vanishing of partition functions

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Motivation

Our original goal was to give a derivation of the Freed-Witten anomaly cancellation condition

$$[H_3] + W_3 = 0$$

(where $H_3 = dB_2$) that was easily applicable to non-perturbative string theory backgrounds, in particular the S-fold backgrounds studied in [Etheredge, IGE, Heidenreich, Rauch '23].

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Since it is often the case that the anomalies of QFTs are much easier to understand than the QFT itself, this allowed us to prove generalised Freed-Witten anomaly cancellation conditions without having to (or being able to) compute the partition function itself.

Example: the free fermion in $d = 4$

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Classically this theory has a symmetry group $U(1)_A \times U(1)_V$, acting as

$$U(1)_V \colon \psi \to e^{2\pi i\alpha}\psi$$
$$U(1)_A \colon \psi \to e^{2\pi i\gamma^5\beta}\psi\,.$$

The axial anomaly

This symmetry group has a 't Hooft anomaly, leading to a non-conservation law for the axial current

$$dj_A = \hat{A}(T\mathcal{M}^4)\,\mathrm{ch}(F_V)\Big|_4 \,.$$

(I will be setting $A_A = 0$, and have redefined a factor of 2 away.) Here $\hat{A}(T\mathcal{M}^4) = 1 - \frac{1}{24}p_1(T\mathcal{M}^4) + \ldots$ and $\mathrm{ch}(F_V)$ is the Chern character.

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Since $j_A^\mu \sim \bar{\psi}\gamma^5\gamma^\mu\psi$ involves ψ, the field being integrated over, this is an operator equation, or in the path integral context a (modified) Ward-Takahashi identity.

The axial anomaly (continued)

Consider an insertion of the operator

$$\mathcal{Q}_V := \int_{\mathcal{M}^4} dj_A$$

in the path integral. Due to the Stokes identity, and the fact that $\partial\mathcal{M}^4 = 0$ (since $\mathcal{M}^4$ is closed by assumption), we have

$$\int [D\psi][D\bar{\psi}]\mathcal{Q}_V e^{-S[\psi,\bar{\psi},A_V]} = 0\,.$$

The axial anomaly (continued)

On the other hand, due to the modified Ward-Takahashi identity

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we have

$$\begin{aligned}\int [D\psi][D\bar{\psi}]\mathcal{Q}_V e^{-S[\psi,\bar{\psi},A_V]} &= \int [D\psi][D\bar{\psi}]\left(\int_{\mathcal{M}^4} dj_A\right) e^{-S[\psi,\bar{\psi},A_V]} \\ &= Z[A_V]\int_{\mathcal{M}^4} \hat{A}(T\mathcal{M}^4)\,\mathrm{ch}(F_V)\,d^4x\,,\end{aligned}$$

with

$$Z[A_V] \coloneqq \int [D\psi][D\bar{\psi}]e^{-S[\psi,\bar{\psi},A_V]}\,.$$

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$$Z[A_V] \coloneqq \int [D\psi][D\bar{\psi}]e^{-S[\psi,\bar{\psi},A_V]}\,.$$

So we learn that

$$Z[A_V]\int_{\mathcal{M}^4} \hat{A}(T\mathcal{M}^4)\,\mathrm{ch}(F_V)\,d^4x = 0\,.$$

The index theorem

What saves the day is the index theorem, which in this case reads

$$\mathrm{ind}(\not{D}_{A_V}) \coloneqq n_+ - n_- = \int_{M^4} \hat{A}(T\mathcal{M}^4)\,\mathrm{ch}(F_V)\,d^4x\,.$$

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The index theorem implies that whenever

$$\int_{M^4} \hat{A}(T\mathcal{M}^4)\,\mathrm{ch}(F_V)\,d^4x \neq 0$$

there's at least one fermionic zero mode, and therefore

$$Z[A_V] = 0\,.$$

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To start going in this direction, notice that the Ward-Takahashi identity

$$dj_A = \hat{A}(T\mathcal{M}^4)\,\mathrm{ch}(F_V)\Big|_4$$

says that the background fields A_V and the curvature are charged under $U(1)_A$, and

$$\mathcal{Q}_A = \int_{\mathcal{M}^4} dj_A$$

measures the total $U(1)_A$ charge of our operator insertions on $\mathcal{M}^4$.

Vanishing from anomalies (continued)

So our conclusion

$$\int [D\psi][D\bar{\psi}] \left(\int_{\mathcal{M}^4} dj_A \right) e^{-S[\psi,\bar{\psi},A_V]} \neq 0 \implies Z[A_v] = 0$$

can be understood as saying that the path integral on the closed manifold $\mathcal{M}^4$ vanishes in the presence of insertions with net $U(1)_A$ charge.

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This follows from a fairly familiar argument, with an easy proof. Consider a current j which is conserved away from operator insertions $\mathcal{O}_1(x_1), \ldots, \mathcal{O}_k(x_k)$:

$$dj = \sum_{i=1}^{k} q_i \delta(x_i) \,.$$

Symmetries as topological operators

Thanks to the identity $dj = \sum_{i=1}^{k} q_i \delta(x_i)$ the operator

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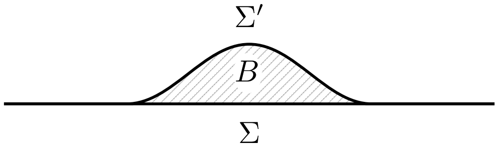

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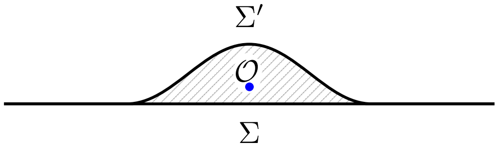

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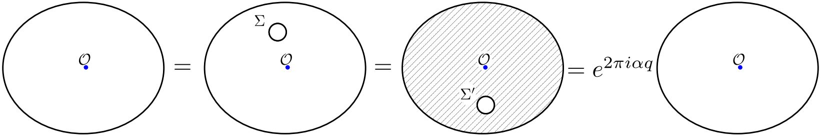
Vanishing from anomalies (continued)

Then, by the following manouver



we can show

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_k(x_k) \rangle = e^{2\pi i \alpha \sum_i q_i} \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_k(x_k) \rangle$$

with

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_k(x_k) \rangle \coloneqq \int [d\Phi] \mathcal{O}_1(x_1) \cdots \mathcal{O}_k(x_k) e^{-S[\Phi]} \,.$$

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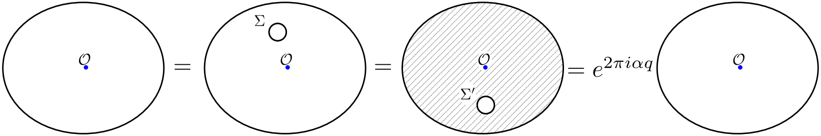

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Since we get to choose α, we have $\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_k(x_k) \rangle = 0$ whenever $\sum_i q_i = 0$.

Rigid transformations

In the context of the free Dirac fermion, this is the statement that under $U(1)_A$ transformations the partition function transforms as

$$Z[A_V] \to e^{2\pi i\alpha\, \mathrm{ind}(\not{D}_{A_V})} Z[A_V]\,.$$

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$$Z[A_V, A_A = 0] \to e^{2\pi i\int_{\mathcal{M}^4}\alpha(x)\hat{A}(T\mathcal{M}^4)\,\mathrm{ch}(F_V)\,d^4x} Z[A_V, A_A = d\alpha]$$

which does not really tell us much about vanishing of $Z[A_V]$. In particular it is easy to make the prefactor non-vanishing (choose a bump form) even if the index vanishes.

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For lack of a better name, we will refer to these transformations with $d\alpha = 0$ as “rigid” transformations.

Generating rigid transformations

How do we represent rigid transformations in terms of topological operators?

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Consider again our operator

$$U_\alpha(\Sigma) \coloneqq \exp\left(2\pi i\alpha \int_\Sigma j\right) .$$

Choosing some Θ such that $\partial\Theta = \Sigma$, we can also write this as

$$U_\alpha(\Theta) = \exp\left(2\pi i\alpha \int_\Theta dj\right) ,$$

which thanks to the modified Ward-Takahashi identity

$$dj = \sum_i q_i \delta(x_i)$$

dresses every operator $\mathcal{O}(x_i)$ with $x_i \in \Theta$ with a phase $e^{2\pi i q_i \alpha}$.

Generating rigid transformations (continued)

So if we want to act on all operators, we can choose $\Theta = \mathcal{M}^4$, and our rigid symmetry generator becomes

$$R_\alpha \coloneqq \exp\left(2\pi i\alpha \int_{\mathcal{M}^4} dj\right) .$$

Rigid transformations in the anomaly theory

This is almost in a form that we can use for generalisations, but there is one extra step that is useful: going to the anomaly theory. (Anomaly theories were nicely reviewed in Matilda's talk on Tuesday, and also appeared in Miguel's talk, so I can skip the detailed review.)

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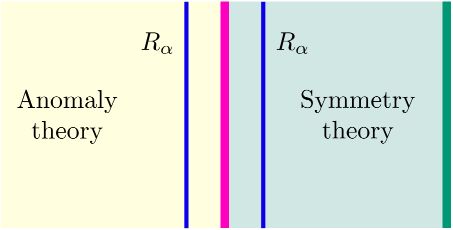
Coming back to our concrete example of the free fermion in $d = 4$, the anomaly theory can be written as the classical theory with action

$$S_{\text{anomaly}} = 2\pi i \int_{X^5} A_A \hat{A}(T\mathcal{M}^4)\,\mathrm{ch}(F_V)$$

again omitting terms that become unimportant when $A_A|_{\mathcal{M}^4} = 0$.

Symmetry generators through the looking glass

It is convenient to use the SymTFT picture to understand how rigid transformations act on the boundary:



Sketchily, the SymTFT is a theory of BF type, with

$$S_{\mathsf{SymTFT}} = 2\pi i \int AdB + \ldots$$

with A taking Dirichlet boundary conditions on the interface between the SymTFT and the anomaly theory, and the holonomy $\exp(2\pi i\alpha \int_\Sigma B)$ representing the standard symmetry generator $U_\alpha(\Sigma) = \exp(2\pi i\alpha \int_\Sigma j)$ in the SymTFT.

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The rigid transformation $U_\alpha(\Theta)$ is then represented by an insertion $\exp\left(2\pi i\alpha\int_\Theta dB\right)$ in the SymTFT path integral, but this is the same as shifting $A \to A + \alpha\delta(\Theta)$.

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The mirror effect of the $A_A \to A_A + \alpha\delta(\Theta)$ on the anomaly theory is precisely as expected:

$$
\begin{aligned}
S_{\text{anomaly}} &= 2\pi i \int_{X^5} A_A \hat{A}(T\mathcal{M}^4)\,\mathrm{ch}(F_V) \\
&\quad \to S_{\text{anomaly}} + 2\pi i\alpha \int_{\mathcal{M}^4} \hat{A}(T\mathcal{M}^4)\,\mathrm{ch}(F_V)
\end{aligned}
$$

using $\Theta = \mathcal{M}^4$ in this case.

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The anomaly arising from a rigid transformation associated to a cycle $\Theta = \mathcal{M}^4$ (with coefficient α) is given by the anomaly theory evaluated on a mapping torus $[0,1]_\tau \times \mathcal{M}^4$, with background fields

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If $S_{\text{anomaly}} \neq 0$ on this background, we have

$$Z[A_V] = e^{-S_{\text{anomaly}}} Z[A_V] \quad \text{with} \quad e^{-S_{\text{anomaly}}} \neq 1$$

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Note that this criterion requires only knowledge of S_{anomaly}, and not of the field theory itself.

$d = 2$ and bosonisation

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If we bosonise the problem, we have the theory of a periodic scalar, which has momentum and winding $U(1)$ symmetries, with a mixed anomaly between them. The action of the scalar coupled to a momentum (say) background is

$$S_\phi = \int (d\phi - A_p) \wedge \star(d\phi - A_p)$$

and the vanishing occurs whenever $[dA_p] \neq 0 \in H^2(\Sigma^2; \mathbb{Z})$.

An important subtlety

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- This answer is required from compatibility with fermionization.
- This answer is required by T-duality:

$$S_{\tilde{\phi}} = \int d\tilde{\phi} \wedge \star d\tilde{\phi} + 2\pi i \int \tilde{\phi} \wedge dA_p \,.$$

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We can alternatively compute the anomaly theory on a mapping torus with an appropriate rigid transformation, with the same result.

Maxwell theory

We can now generalise the periodic scalar in two dimensions to the Maxwell theory in four dimensions (relevant to the case of the D3 brane, for instance). This is a theory with $U(1)_e^{[1]} \times U(1)_m^{[1]}$ symmetry. The action, for topologically trivial B_e, is

$$S_{\mathsf{Maxwell}} = \frac{1}{2e^2} \int (da - B_e) \wedge \star (da - B_e) + 2\pi i \int B_m \wedge (da - B_e)\,,$$

and there is an anomaly

$$S_{\mathsf{anomaly}} = 2\pi i \int dB_e \wedge B_m\,.$$

Vanishing of the partition function for Maxwell

Rigid transformations for $U(1)_m$ are associated with 3-cycles Θ (over which we integrate df, with $f := da$), with Poincaré dual $\Pi^1 \in Z^1(\mathcal{M}^4)$.

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Consider the mapping torus $[0,1]_\tau \times \mathcal{M}^4$, with fields

$$B_e(\tau, x \in \mathcal{M}^4) = B_e(x)$$
$$B_m = \alpha d\tau \wedge \Pi^1$$

encoding a rigid $U(1)_m^{[1]}$ transformation with parameter α.

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$$Z_{\text{anomaly}} = e^{-2\pi i \int_{[0,1]\times\mathcal{M}^4} B_m \wedge dB_e} = e^{-2\pi i \alpha \int_{\mathcal{M}^4} \Pi^1 \wedge dB_e} .$$

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For the partition function $Z_{\text{Maxwell}}[B_e]$ not to vanish we need $Z_{\text{anomaly}} = 1$ for all Π^1 and α, which requires $[dB_e] = 0$.

Other applications

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In the paper we discuss a number of examples:

- The actual Freed-Witten anomaly (including the W_3 term).
- Non-abelian extensions, as in [Kapustin '99].
- Theories with anomaly theories of Dijkgraaf-Witten type.
- Branes on S-folds.
- M5 branes.
- Vanishing due to mod-2 indices in 2d.
- 3d minimal TQFTs.

Back to the $d = 4$ Dirac fermion

Consider the theory of the Dirac fermion, where additionally we gauge the $U(1)_V$ symmetry by path integrating over the connection a_V.

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As discussed in Fabio's talk, the modern perspective is that this leads to non-invertible axial symmetry operators, rather than just absence of axial symmetry.

A naive attempt

We could try to define conserved currents of the form

$$U_\alpha(\Sigma) = \exp\left(2\pi i\alpha \int_\Sigma j_A - a_V \wedge da_V\right) .$$

If α is a generic real number, this has two (related, but distinct) potential issues:

- The operator is not well defined as a function of Σ, since the $\alpha \int_\Sigma a_V \wedge da_V$ term is not invariant under the gauge transformations we encounter when changing patch.

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- The exponent is not invariant under large gauge transformations whenever Σ has rich enough topology to support them.

A naive attempt (continued)

The precise problem is the lack of invariance of

$$U_\alpha(\Sigma) = \exp\left(2\pi i\alpha \int_\Sigma j_A - a_V \wedge da_V\right)$$

under large gauge transformations of a_V: if $a_V \to a_V + \lambda$, with $\lambda \in \Omega^1_{\mathbb{Z}}(\mathcal{M}^4)$, we have

$$U_\alpha(\Sigma) \to e^{-2\pi i\alpha \int_\Sigma [\lambda]\wedge[da_V]} U_\alpha(\Sigma)\,.$$

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(Perhaps the best way to think about this is in terms of the lattice, where this is an instance of Elitzur's theorem. [Elitzur '75]
Alternatively, we can think of overaging the operator over the gauge orbit [Karasik '22], [IGE, Iqbal '22], [Arbalestrier, Argurio, Tizzano '24])

An alternative proposal

An alternative set of operators, proposed in [Choi, Lam, Shao '22], [Córdova, Ohmori '22], replace the problematic piece

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Since this partition function involves a path integral in 3d, it leads to non-invertibility of the resulting operator.

Vanishing of the $\mathcal{A}^{N,p}$ partition function

(following [Karasik '22])

Nevertheless, this (standard and technically correct) presentation of the situation is in my opinion incomplete in a crucial way, due to an observation originally in [Karasik '22].

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More or less by definition, the $\mathcal{A}^{N,p}$ theory has an anomaly of the form

$$S_{\text{anomaly}} = \frac{2\pi i p}{2N} \int B_2 \wedge B_2 \,.$$

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We evaluate the anomaly theory on a mapping torus $[0,1]_\tau \times \Sigma$ with

$$B_2(\tau, x \in \Sigma) = B_2(x) + d\tau \wedge \lambda$$

with $[\lambda] \in H^1(\Sigma; \mathbb{Z}_N)$, corresponding to the action of the rigid transformation λ ($\mathbb{Z}_N$ valued because we have a $\mathbb{Z}_N$ 1-form symmetry).

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We obtain

$$S_{\text{anomaly}} = \frac{2\pi i p}{N} \int_{\Sigma} [\lambda] \wedge [da_V] \,.$$

The partition function $Z_{\mathcal{A}^{N,p}}[B_2 = da_V \pmod N]$, and therefore the non-invertible operator at hand, vanishes unless

$$Z_{\text{anomaly}} = e^{-S_{\text{anomaly}}} = 1$$

for all λ. Equivalently, we need

$$\left[\frac{p}{N} da_V\right] \in H^2_{dR}(\Sigma)_{\mathbb{Z}} \,,$$

which was precisely the condition we found for gauge invariance in the naive analysis! (with $\alpha = p/N$)

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That is, the non-invertible operator based on $\mathcal{A}^{N,p}$ is only non-zero whenever the naive operator is well defined anyway.

Conclusions

The method I just presented is a fairly hands-on method for proving the vanishing of partition functions in situations where the anomaly theory is known, even if the theory itself is poorly understood.

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One interesting application is to the physics of branes, in the form of generalised Freed-Witten anomaly cancellation conditions.

There are interesting directions to explore further, for instance:

- Vanishing conditions from more general categorical symmetries?
- Abelian subgroups of non-abelian symmetries (for example relating to the results in [Meynet, Migliorati, Savelli, Tortora '25]).