CONTINUITY ON THE BOUNDARY OF MODULI SPACE

Severin Lüst

LPTHE

CNRS / Sorbonne Université

Strings and Geometry 2026

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[hep-th/2606.xxxxx]
with Paul Balavoine

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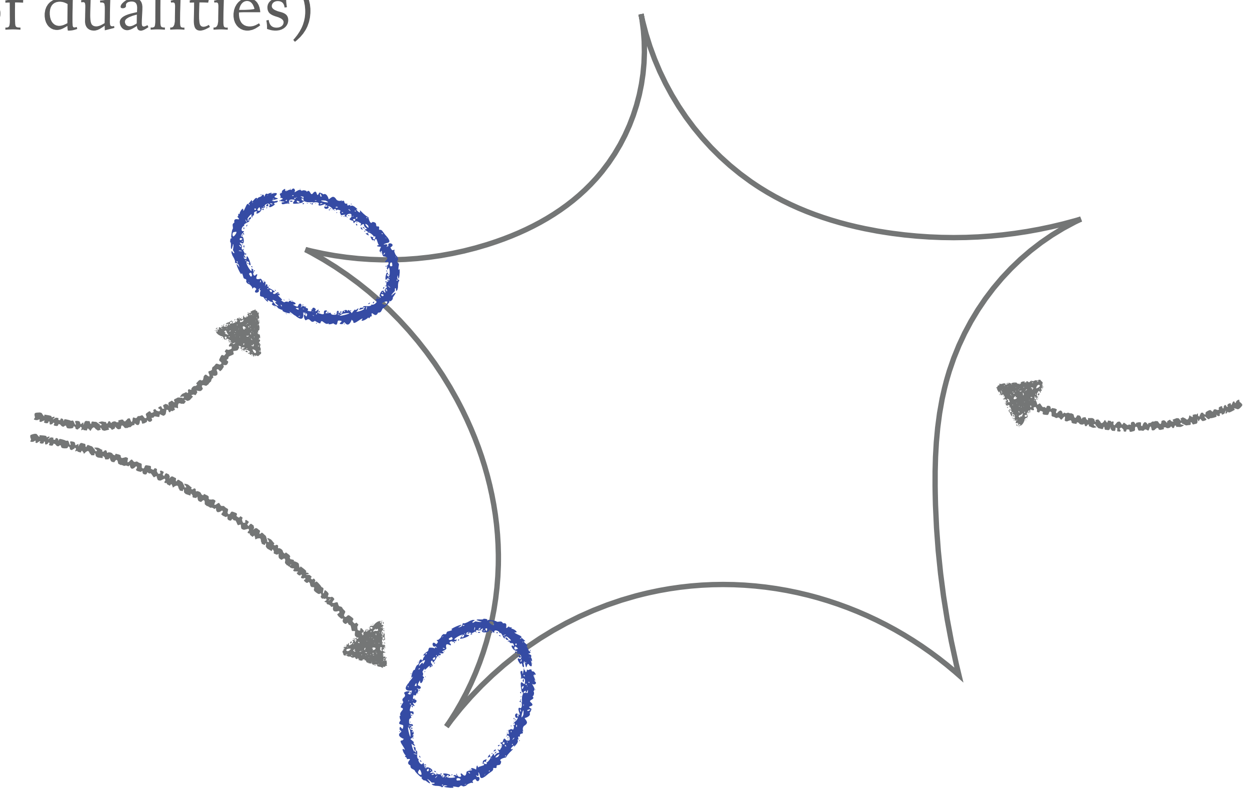
GLOBAL BOUNDARY OF THE MODULI SPACE

- Cartoon of the moduli space:

  (c.f. web of dualities)

asymptotic
limits / boundaries:
well studied
(Swampland program)



What lies "in between" the
asymptotic boundaries?

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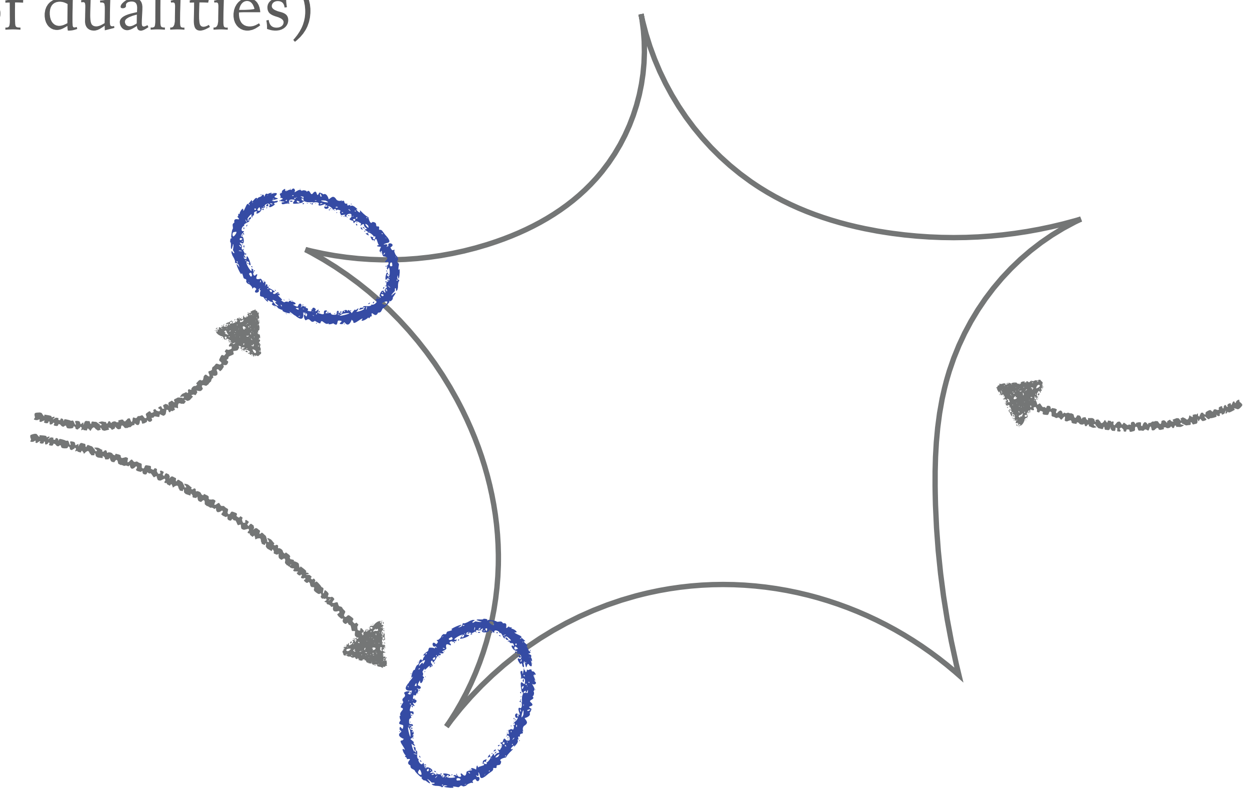

What lies "in between" the
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→ What is the global structure of the boundary of the moduli space?

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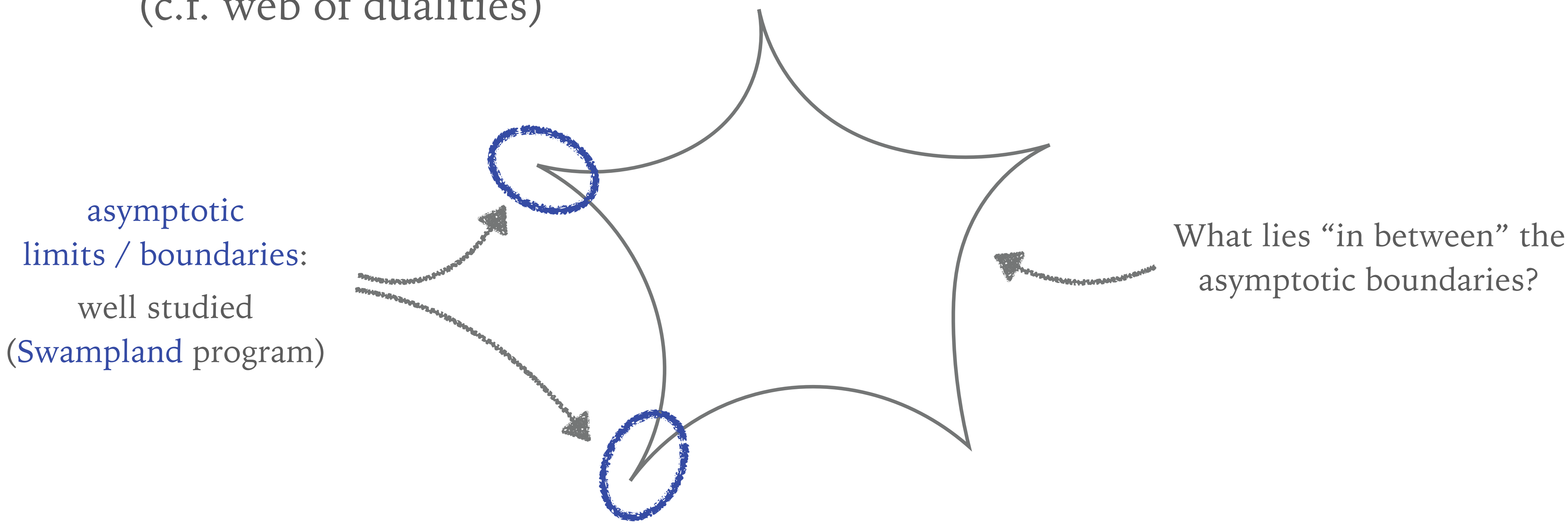

→ What is the global structure of the boundary of the moduli space?

c.f. marked moduli space:

$$\mathscr{M} = \frac{\mathscr{T}}{\Gamma}$$

$\mathscr{T}$ ← marked moduli space

Γ ← duality group

[Raman, Vafa '24][Delgado, van de Heisteeg, Raman, Torres, Vafa, Xu '24]

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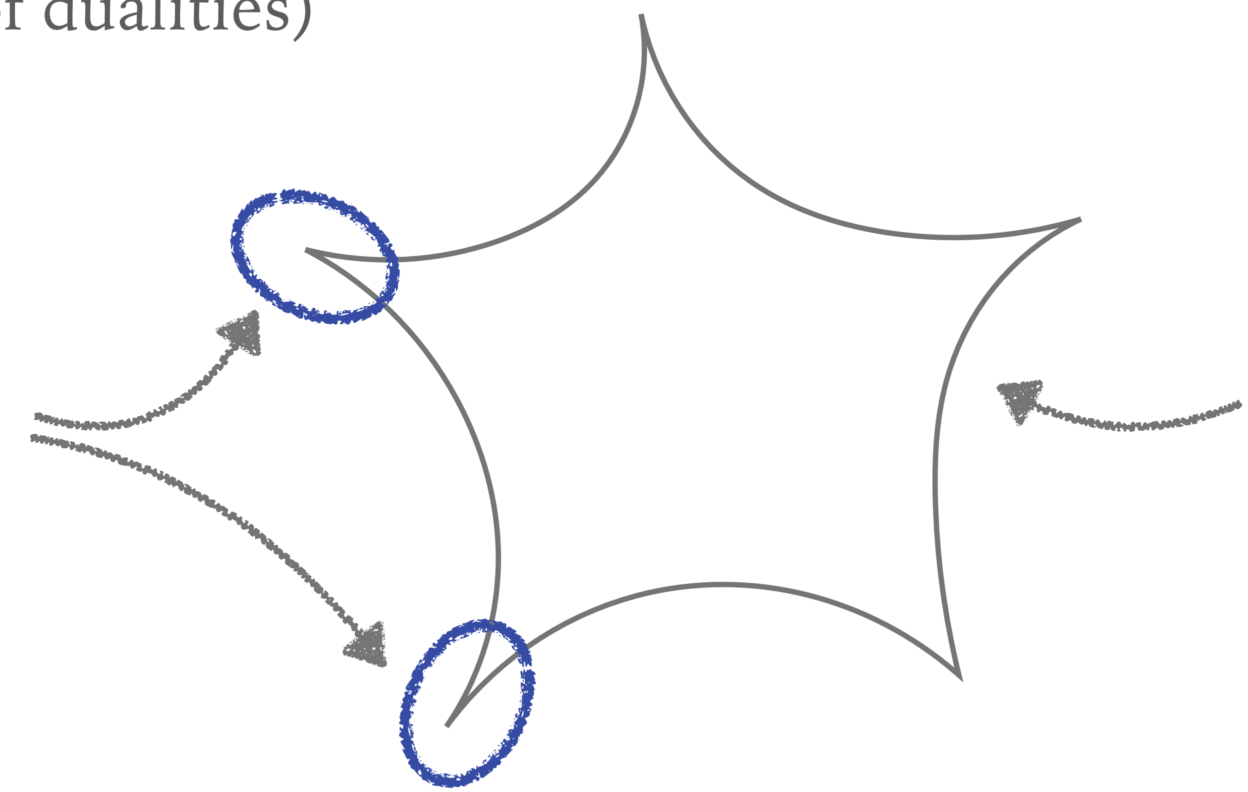

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marked moduli space ($\mathscr{T}$)

duality group (Γ)

here → *structure / continuity on $\partial\mathscr{T}$*

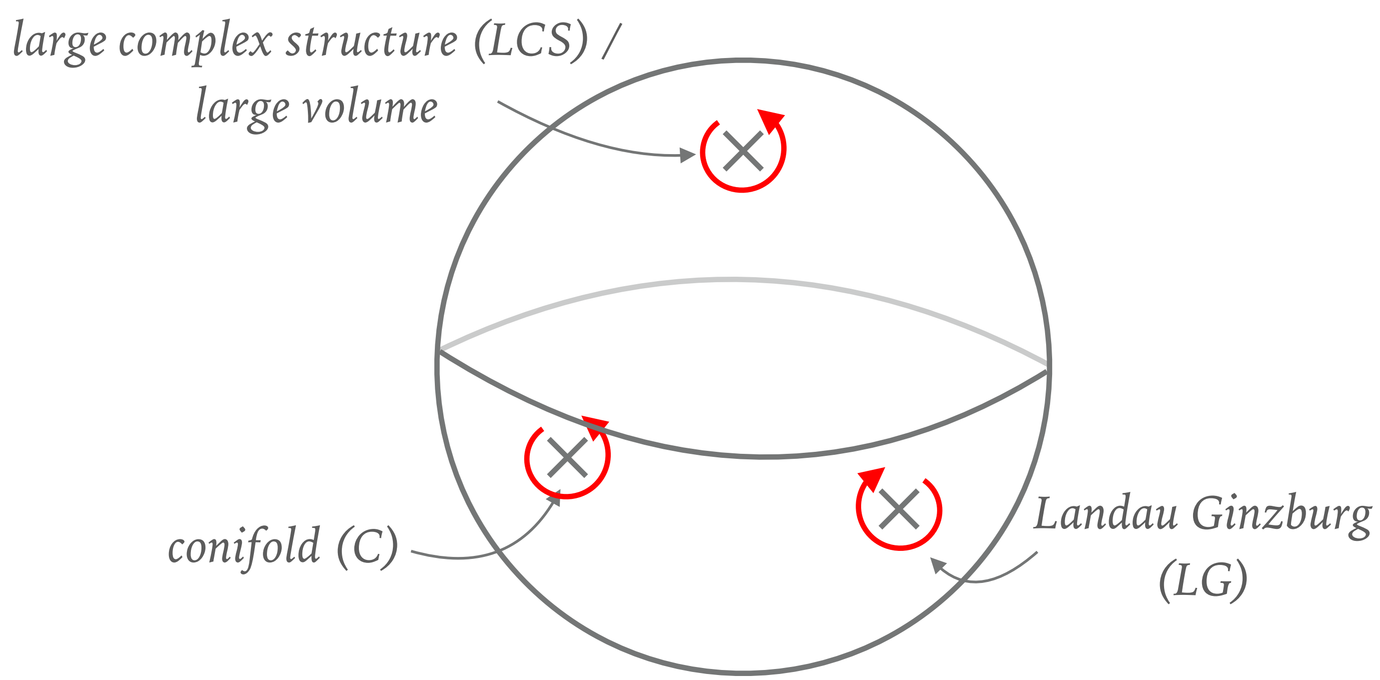
[Raman, Vafa '24][Delgado, van de Heisteeg, Raman, Torres, Vafa, Xu '24]

COVERING SPACE OF THE MODULI SPACE:

- Calabi-Yau compactifications: [Candelas, de la Ossa, Green, Parkes '91]
  one-parameter vector-multiplet moduli space (e.g. mirror quintic):

$$\mathscr{M} = S^2 \backslash \{z_i\}$$

(punctured sphere)



  - boundary: $\{z_{LCS}, z_C\}$
  - period maps $\Pi^I(z)$:
    monodromies around punctures z_i

    → *monodromy group* Γ

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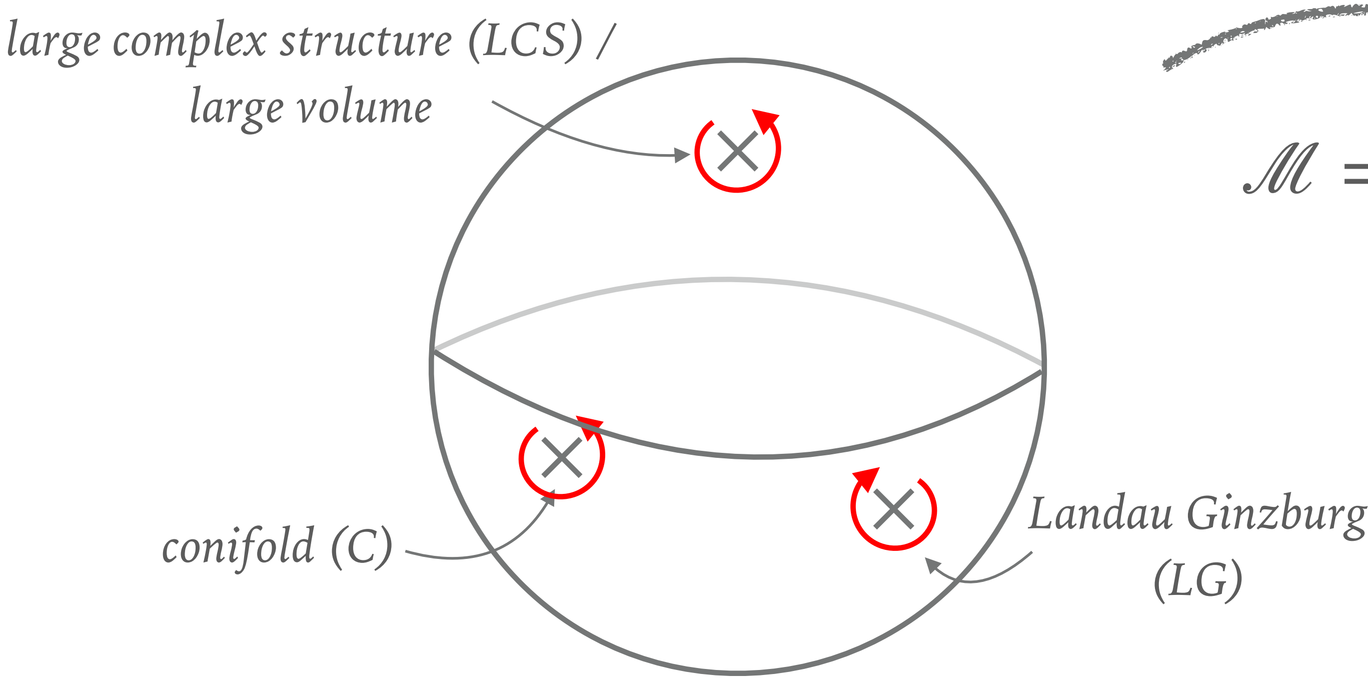
(punctured sphere)

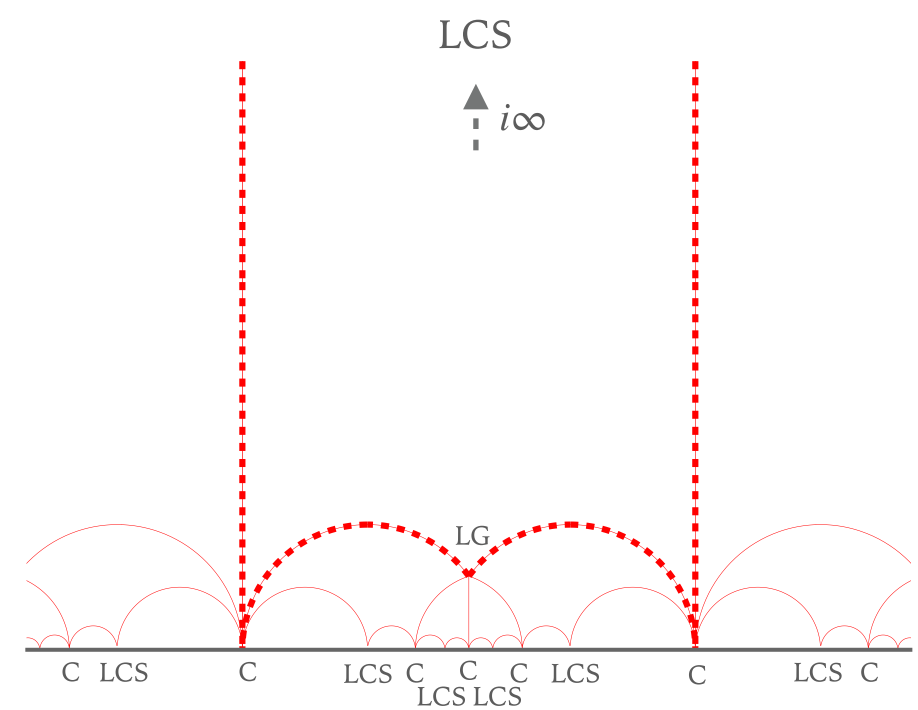
(universal) covering space

$\mathbb{H}$

(upper half-plane)



$$\mathcal{M} = \frac{\mathbb{H}}{\Gamma}$$



- ➤ boundary: $\{z_{LCS}, z_C\}$
- ➤ period maps $\Pi^I(z)$:
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- ➤ periods: single valued
- ➤ boundary: $\partial\mathbb{H} = S^1$

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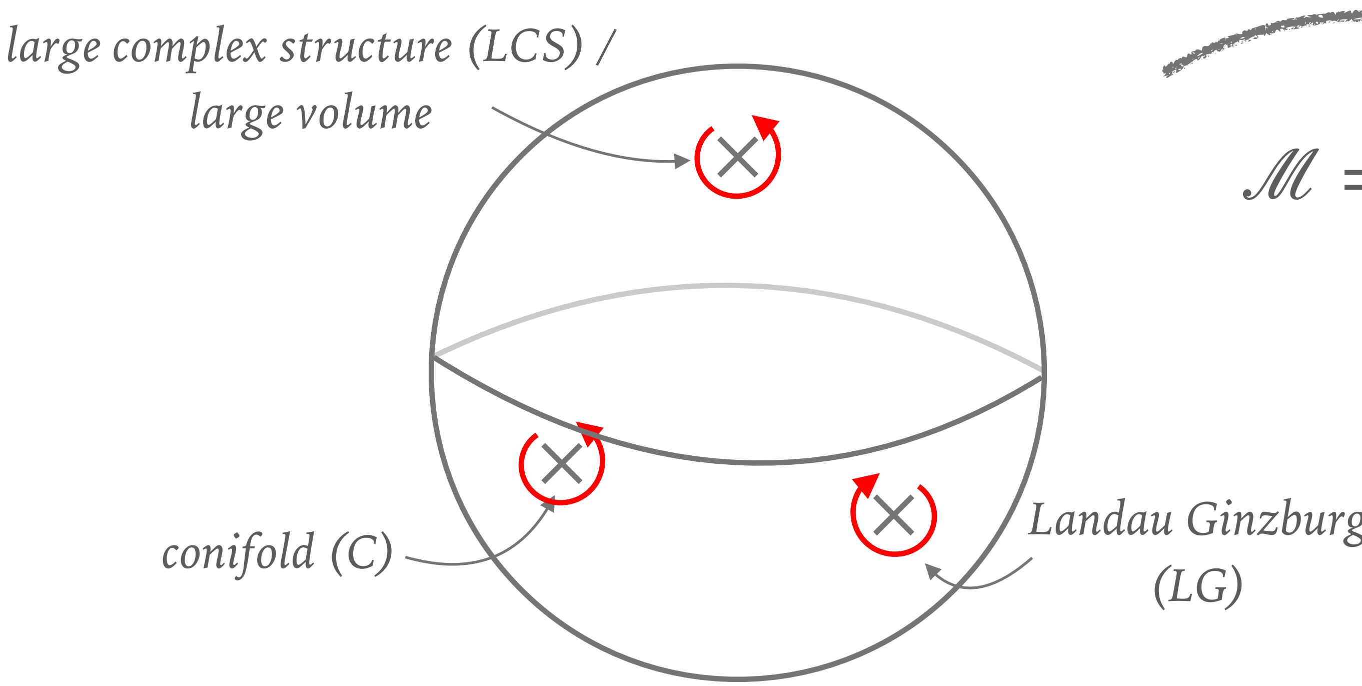

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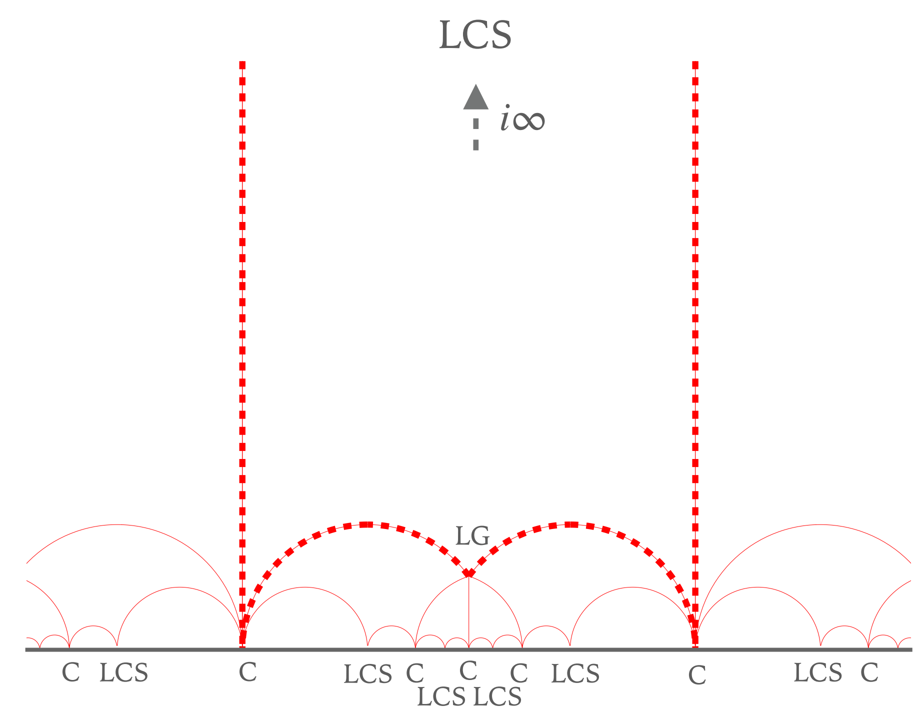

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- periods: single valued
- boundary: $\partial\mathbb{H} = S^1$

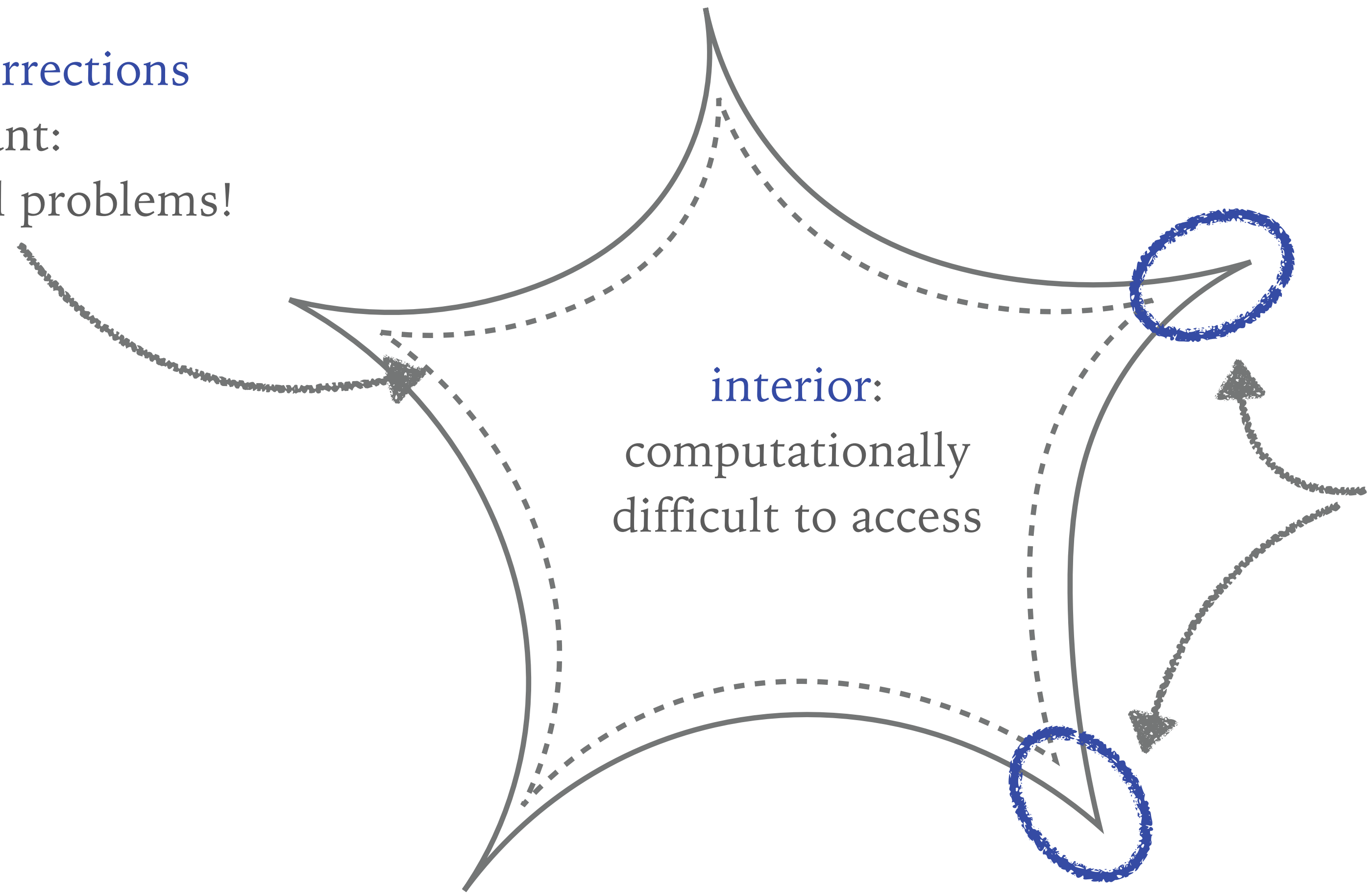
Question:

Do periods extend continuously to $\partial\mathbb{H}$?

MOTIVATION: INFORMATION ON THE INTERIOR

Cartoon of the moduli space:

string / quantum corrections
become relevant:
Dine-Seiberg / control problems!



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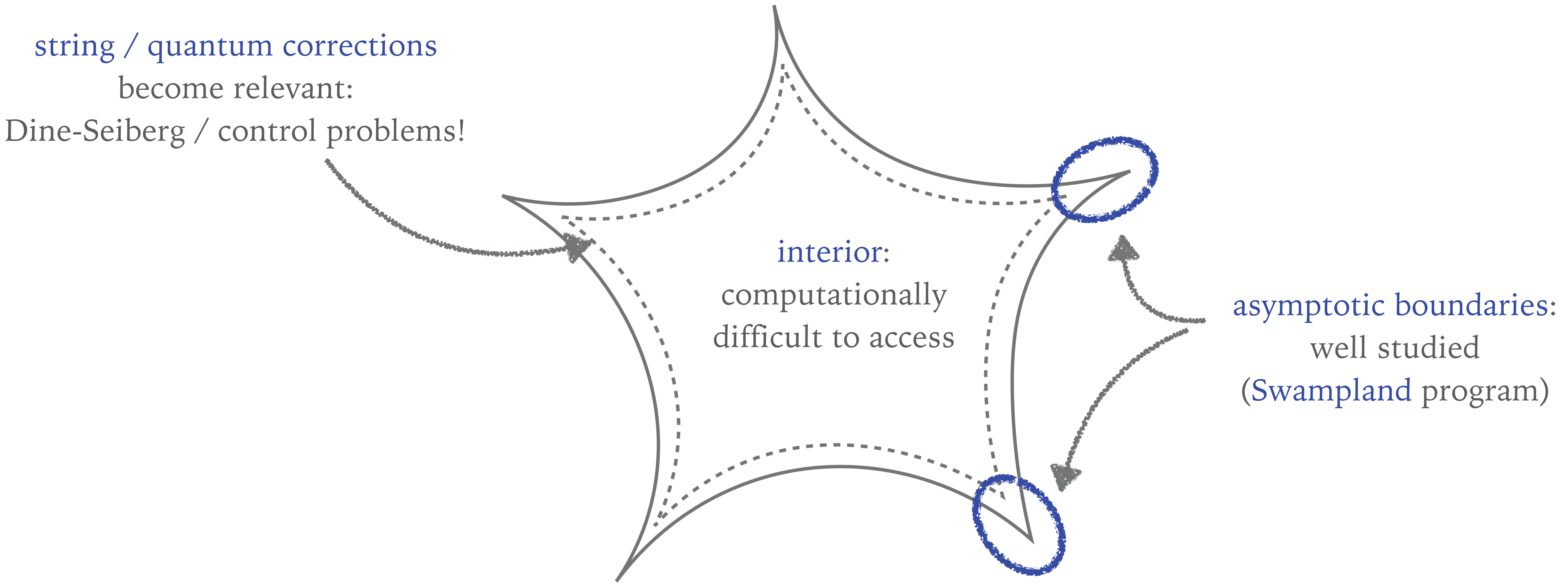

Question:

(global structure of the)
asymptotic boundary

MOTIVATION: INFORMATION ON THE INTERIOR

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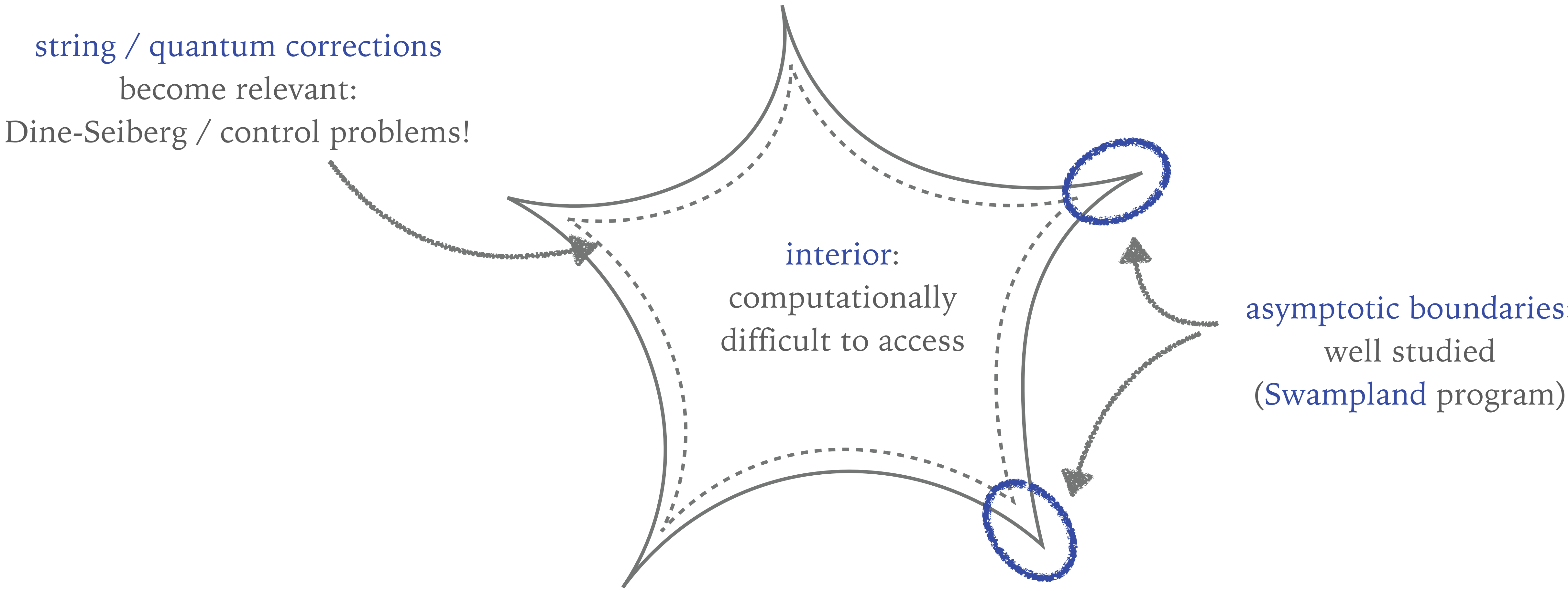


Question:

(global structure of the) asymptotic boundary → information on the interior of moduli space?

INDEX FOR FLUX VACUA

[hep-th/2405.04584]

CLASSICAL FLUX VACUA OF IIB / M-THEORY

- IIB / M-theory on Calabi-Yau n-fold (IIB: $n = 3$; M-theory: $n = 4$)
- Flux compactification:

$h^{n-1,1}$ complex str. moduli (volumes of n-cycles) $\longrightarrow$ stabilized by $G_n \neq 0$ (n-form flux)

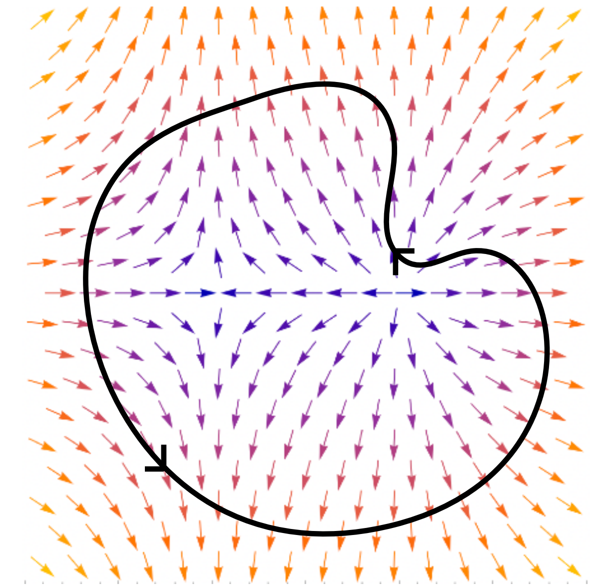
[Gukov, Vafa, Witten '99]

- Superpotential: $$W(z) = \int_{CY_n} G_n \wedge \Omega(z) = g_I \Pi^I(z)$$ (Π^I: period maps)
- F-term condition: $D_i W = 0 \quad \Leftrightarrow \quad \star\, G_n = (-i)^n G_n$

$\longrightarrow$ classical stabilization of c.s. moduli

(*) $D_i W = 0$ does not include Kähler moduli,
and does not yet guarantee vacuum of the full quantum theory
(see, e.g., [Kachru, Kallosh, Linde, Trivedi '03], [Balasubramanian, Berglund, Conlon, Quevedo '05])

INDEX OF VECTOR FIELDS



- $\overrightarrow{X}$ vector field on an open subset $U \subseteq \mathbb{R}^n$ (differentiable; finitely many, isolated zeros x_i)
- Index / degree of $\overrightarrow{X}$:

$f = \frac{X}{\|X\|}$

$$\mathrm{ind}_U(X) = \deg f = \frac{1}{\mathrm{vol}(S^{n-1})} \int_{\partial U} f^*\omega$$

ω: volume form on S^{n-1}

(see also Poincaré-Hopf theorem)

- Counting of zeros:

$$\mathrm{ind}_U(X) = \sum_i \mathrm{ind}_{U_i}(X)$$

sum over zeros x_i

U_i: small neighborhood of x_i

- Index for flux vacua: $X = DW$ *(as real vector field; for given flux G)*

(if DW is non-singular)

$\mathrm{ind}_{\mathscr{M}}(DW) \neq 0$ → $DW = 0$ → *classical flux vacuum*

(somewhere in the interior of the moduli space $\mathscr{M}$)

ASYMPTOTIC PERIOD MAPS

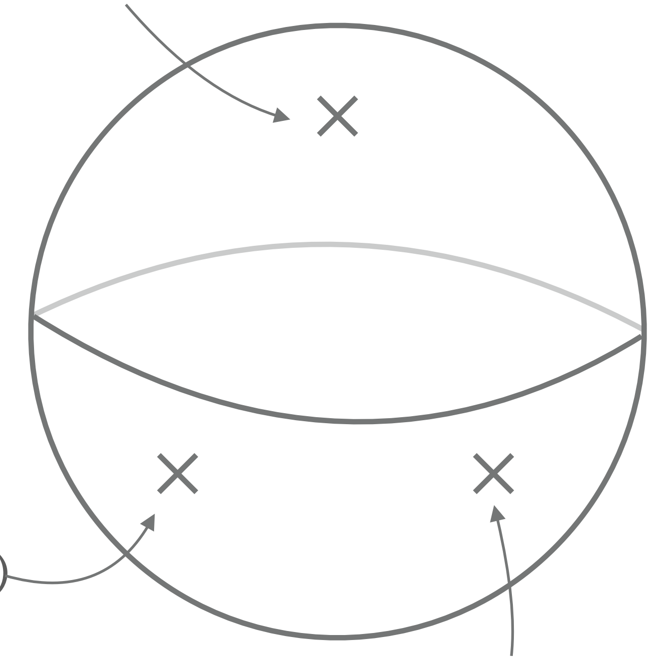
[SL '24]

- Periods as series expansion around critical locii (solve Picard-Fuchs eq. by Frobenius method)

$$W(z) = g_I \Pi^I(z)$$

- Example: Fermat mirror sextic CY 4-fold:
  There are 3 critical locii / degeneration points:
  - $\psi = \infty$: large complex structure point
  - $\psi^6 = 1$: conifold point
  - $\psi = 0$: Landau-Ginzburg / Fermat point

  [Grimm, Ha, Klemm, Klevers '09][van de Heisteeg '24]



- At degeneration points:
  Simple asymptotic expressions for periods:

e.g. large complex structure:

$$t \sim \log \psi \to i\infty$$

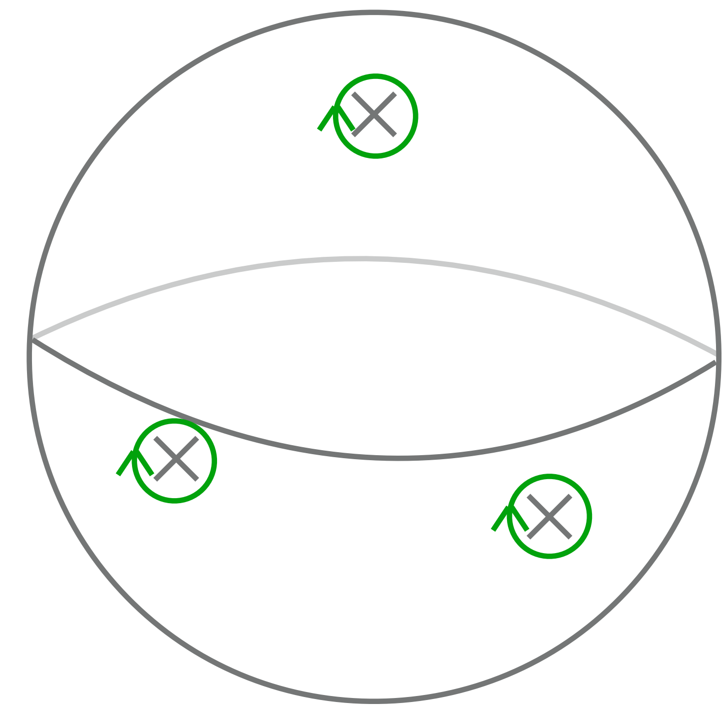
$$\Pi^1 = 1 + \mathcal{O}(e^{2\pi i t})$$
$$\Pi^2 = t + \mathcal{O}(e^{2\pi i t})$$
$$\vdots$$
$$\Pi^5 = \frac{1}{4}t^4 + \frac{15}{8}t^2 - \frac{105\zeta(3)i}{2\pi^3}t - \frac{75}{64} + \mathcal{O}(e^{2\pi i t})$$

CONTOUR INTEGRAL AROUND DEGENERATION POINTS?

- (Naive) idea: close contour around punctures:

Hope: exact analytic computation possible

(periods allow for simple, asymptotic expansions)



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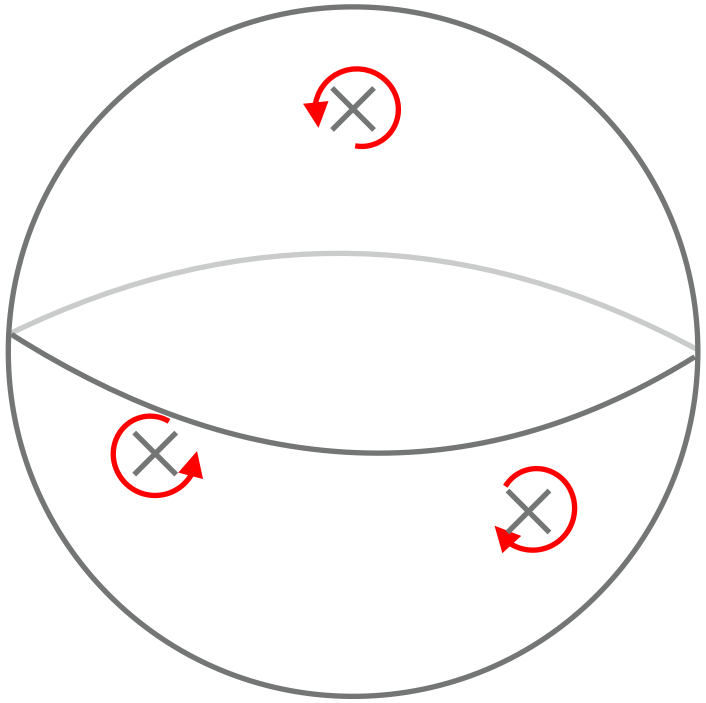

- Problem:
  Periods: monodromies around punctures

→ monodromy group Γ

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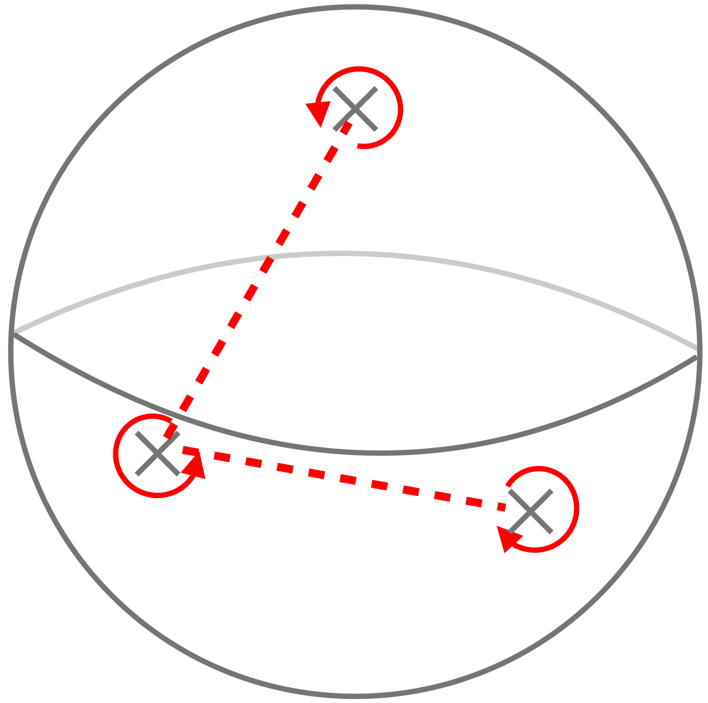

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- Periods / DW not single valued → branch cuts

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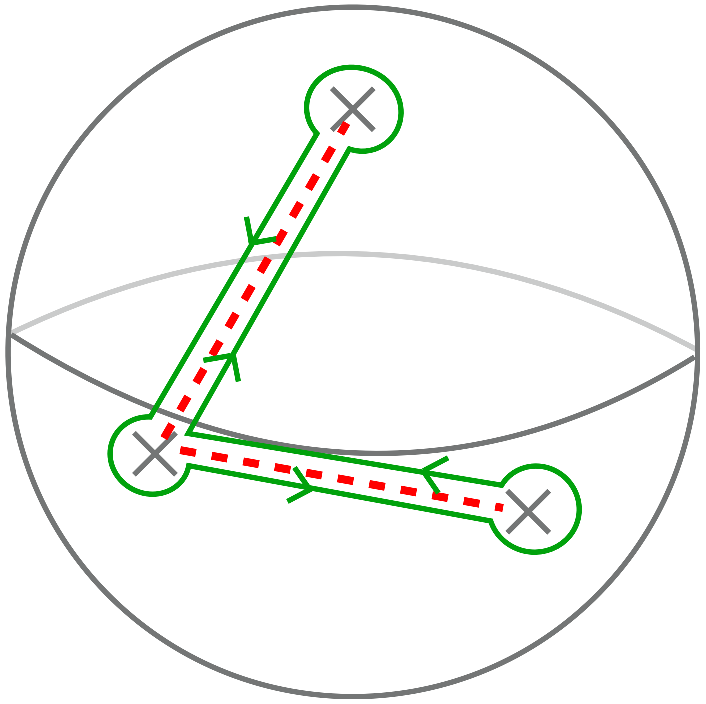

- Problem:
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- Periods / DW not single valued → branch cuts

→ contour must enclose also branch cuts! [SL '24]

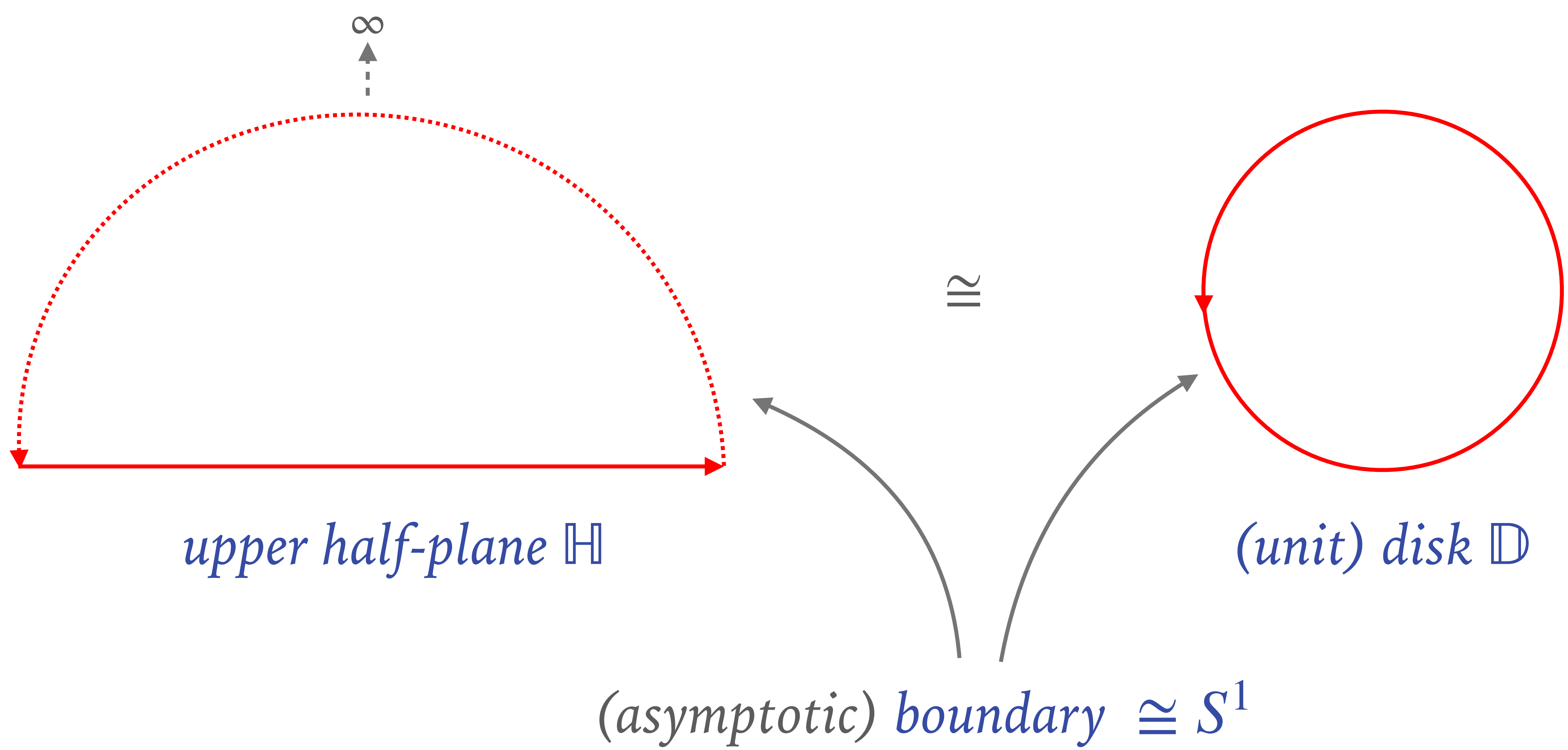
(and cannot be closed at asymptotic boundary)

THE BOUNDARY OF THE COVERING SPACE

THE COVERING SPACE

[P. Balavoine, SL, *to appear*]

- **Problem:** branch cuts probe the interior of the moduli space!
- **Solution:** work on the (universal) covering space.
  (c.f. marked moduli space [Raman, Vafa ’24][Delgado, van de Heisteeg, Raman, Torres, Vafa, Xu ’24])
- For one-parameter CY manifolds (e.g., mirror sextic):



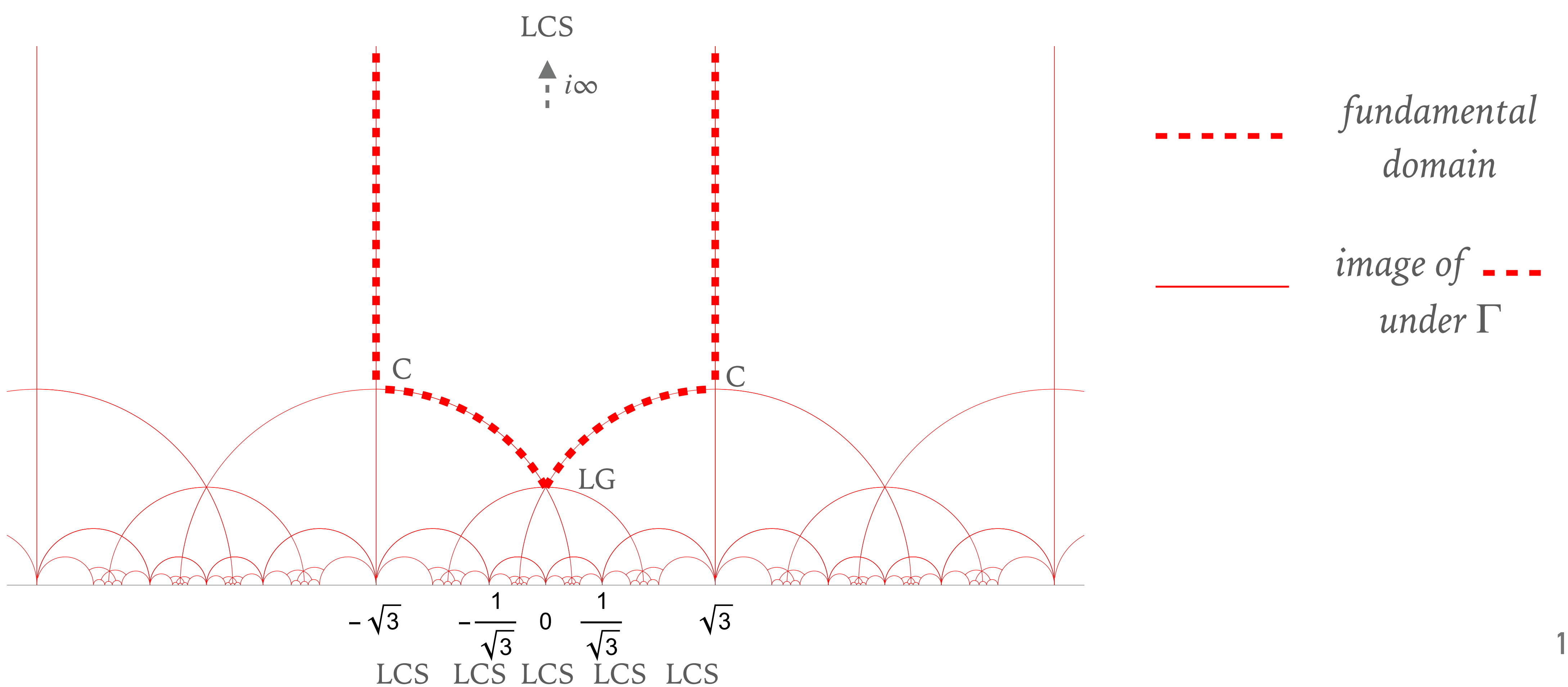
ACTION OF THE MONODROMY GROUP:

[P. Balavoine, SL, *to appear*]

- Identify the monodromy group Γ as a subgroup of $PSL(2,\mathbb{R})$:
- Mirror sextic: (see [Candelas, de la Ossa, Green, Parkes '91] for mirror quintic)

$$\gamma_{LCS} = \begin{pmatrix} 1 & 2\sqrt{3} \\ 0 & 1 \end{pmatrix} \qquad \gamma_{LG} = \frac{1}{2}\begin{pmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{pmatrix} \qquad \gamma_C = \gamma_{LG} \cdot \gamma_{LCS} = \frac{1}{2}\begin{pmatrix} \sqrt{3} & 7 \\ -1 & -\sqrt{3} \end{pmatrix}$$

- Action on the covering space and fundamental domain:

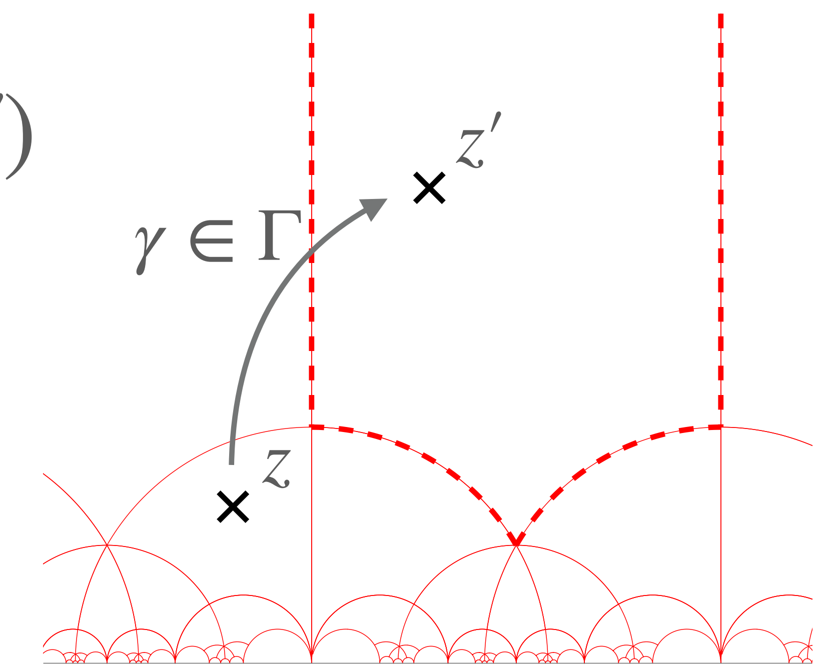


PERIODS ON THE COVERING SPACE

[P. Balavoine, SL, *to appear*]

- Monodromy group Γ: map periods from fundamental region onto full upper half-plane:

$$\Pi^I(z) = M(\gamma)^I{}_J \Pi^J(z')$$



- Γ: Fuchsian group of first kind:

maps LCS point at $i\infty$ densely onto the real line

- Periods on the boundary $\partial\mathbb{H} \cong S^1$:

defined purely algebraically in terms of:

a) monodromy group Γ

b) asymptotic periods at LCS ($\Pi^I \sim t^I$, $I = 0,\ldots,4$)

PERIODS ON THE COVERING SPACE

[P. Balavoine, SL, *to appear*]

➤ Problem: Periods diverge at LCS

$$\textit{for}\quad \frac{\log\psi}{2\pi i}\sim t\to i\infty: \qquad \begin{array}{c}\Pi^I\sim t^I\\ (I=0,\ldots,4)\end{array}$$

➤ Normalized periods:

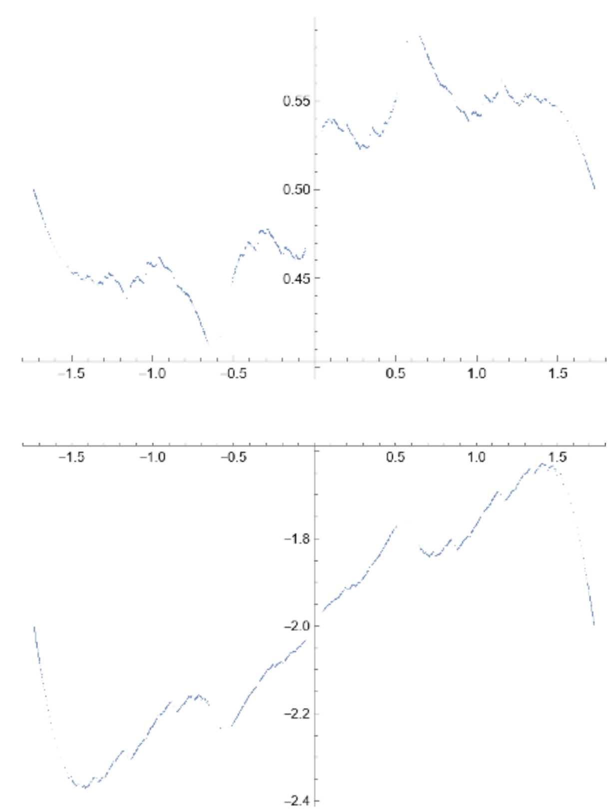
$$\overline{\Pi}^I\equiv\frac{\Pi^I}{\Pi^0}$$

- meromorphic on (open) upper half-plane
- finite on $\partial\mathbb{H}$

normalized periods $\overline{\Pi}^I$ continuos on $\overline{\mathbb{H}}$?

→ *Answer:* ***no!***

(Schwarz reflection principle would imply differentiability on $\partial\mathbb{H}$)



Can we still compute index on $\partial\mathbb{H}$?

A CONTINUOUS FUNCTION ON $\overline{\mathbb{H}}$

[P. Balavoine, SL, *to appear*]

- Normalized Kähler covariant derivatives:

$$F^I = \frac{D\overline{\Pi}^I}{D\overline{\Pi}^0} = \frac{D\Pi^I}{D\Pi^0}$$

Periods of $D\Omega \in H^{3,1}$

- Well-defined and continuous on closure $\overline{\mathbb{H}}$ if
  - Period representation $M(\gamma)^I{}_J$ is relatively Anosov
  - $x \in \partial\mathbb{H}$, $z \in \mathbb{H}$: $F^I(x)\eta_{IJ}F^J(z) \neq 0 \quad (\star)$
    - $(\star)$ follows from non-degeneracy of Hodge product for all CY 4-folds
    - $(\star)$ does not hold for (normalized) periods $\overline{\Pi}^I$
- [Filip '22]: Period representation of mirror sextic is relatively Anosov
- Does not hold for all CY 3-folds [Filip '22] / 4-folds [P. Balavoine, SL, wip]

RELATIVELY ANOSOV REPRESENTATIONS

- Fuchsian group $\Gamma \subset SL(2,\mathbb{R})$ of first kind, [Zhu '21; Zhu, Zimmer '22]
  representation $\gamma : \Gamma \to GL(d,\mathbb{C})$,
  series $(\gamma_n) \subset \Gamma$ such that $\gamma_n \cdot z_0 \to x \in \partial\mathbb{H}$ define:

$$\xi(x) = \lim_{n\to\infty} U_1(\rho(\gamma_n)) \qquad \xi^*(x) = \lim_{n\to\infty} U_{d-1}(\rho(\gamma_n))$$

$U_i(M)$ = space spanned by M's i largest singular directions

- γ is relatively Anosov if it is
  - equivariant: $\xi(\gamma \cdot x) = \rho(\gamma) \cdot \xi(x)$
  - transverse $x \neq y : \xi(x) \oplus \xi^*(y) = \mathbb{C}^d$
  - strong dynamics preserving
- For CY 4-folds (requires rel. Anosov and ★): [P. Balavoine, SL, *to appear*]
  $\xi^I(x)$ is continuous extension of F^I to $\partial\mathbb{H}$

THE INDEX ON THE COVERING SPACE

[P. Balavoine, SL, *to appear*]

- F^I and flux vacua:

continuous on $\overline{\mathbb{H}}$*,*
non-singular on $\mathbb{H}$ → $F^I = \dfrac{D\Pi^I}{D\Pi^0}$ ← *non-zero on* $\mathbb{H}$

fluxes → $g_I F^I(z) = 0$

$$\Leftrightarrow$$

$$DW(z) = g_I D\Pi^I(z) = 0$$

- Index for flux vacua on $\mathbb{H}$:

$$\mathrm{ind}_{\partial\mathbb{H}}(g_I F^I) \neq 0 \qquad \Longrightarrow \qquad DW = 0$$

somewhere on $\mathbb{H}$

- Contour integral along $\partial\mathbb{H} = i\infty \cup \mathbb{R}$:

$$\mathrm{ind}_{\partial\mathbb{H}}(g_I F^I) = \mathrm{ind}_{i\infty}\big(g_I F^I\big) + \sum_{x\in\mathbb{R}:\ g_I F^I(x)=0} \mathrm{ind}_x\big(g_I F^I\big)$$

← *map back to* $i\infty$ *with* Γ*, and use LCS expressions to compute*

possible complication:
what about zeros not in the Γ*-image of* $i\infty$*?*

CONCLUSIONS

- Index defined by contour integral:

  Sufficient condition for flux vacua
  on interior of moduli space

- monodromies of period maps:

  contour can only be closed at asymptotic infinity
  on boundary of covering space

- continuity of normalized $H^{3,1}$-periods:

  relatively Anosov period representation of
  monodromy group

THANK YOU!