On the spectrum of relativistic electrons' beam passing through periodical medium

N. Shul'ga, E. Bulyak

National Science Center "Kharkov Institute of Physics and Technology"

Kharkov, Ukraine

e-mail: shulga@kipt.kharkov.ua

- coherent radiation in crystals
- relativistic electrons' spectrum in undulator
- destruction of coherent effect in undulator radiation

Minsk, 2016

Some directions of our works

- **Coherent Bremsstrahlung and scattering in crystals at high energies**
- Electromagnetic processes with “half-bare” electrons
- Landau-Pomeranchuk-Migdal effect in radiation
- Beam-beam radiation
- Eikonal and WKB approximation
- Dynamical chaos phenomenon
- Electromagnetic showers in crystals
- Beam deflection by bent crystals
- etc.

Coherent length (Ter-Mikaelian, 1953)

$$l_c = 2\varepsilon\varepsilon' / m^2\omega$$

$$d\sigma \approx \int d^2 q_\perp \int_{q_{\min}}^{\infty} dq_\| \frac{q_\perp^2}{q_\|^2} \left|U_q\right|^2$$

$$\varepsilon = \varepsilon' + \omega, \qquad \mathbf{p} = \mathbf{p}' + \mathbf{k} + \mathbf{q}$$

$$l_c = \frac{2\varepsilon\varepsilon'}{m^2\omega}$$

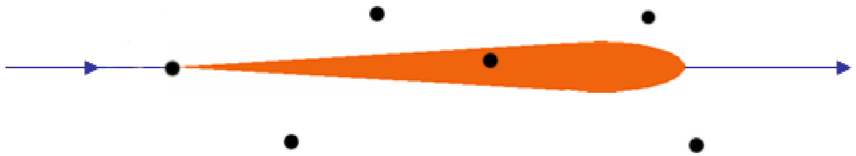
$$r_{\| eff} \approx q_{\| eff}^{-1} \approx l_c = 2\varepsilon\varepsilon' / m^2\omega$$

$$\varepsilon = 100 Mev \quad \omega = 100 kev \quad l_c \approx 10^{-4} cm$$

$$r_{\perp eff} \approx 1 / q_{\perp eff} \approx R$$

$$\varepsilon = 100 Mev \quad \lambda \approx 1 cm \quad l_c \approx 10^3 m!!!$$

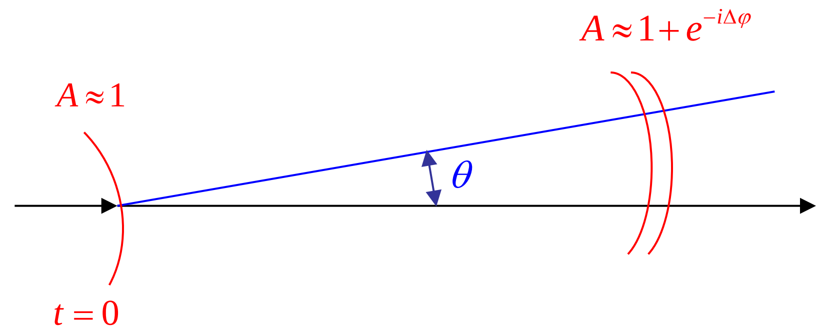
Coherent length (Landau-Pomeranchuk, 1953)



$$\frac{dE}{d\omega\, do} = \frac{e^2}{4\pi^2}\left| \vec{k} \times \int_{-\infty}^{\infty} dt\, \vec{\upsilon}(t)\, e^{i\left(\omega t - \vec{k}\vec{r}(t)\right)} \right|^2$$

$$\vec{\mathrm{v}}(\mathrm{t}) \approx \vec{\mathrm{v}}_0 \cdot \left(1 - \frac{1}{2}\mathrm{v}_\perp{}^2(t)\right) + \vec{\mathrm{v}}_\perp(\mathrm{t})$$

$$\Delta\varphi = \omega\Delta t - \vec{k}\vec{r}\left(\Delta t\right) <\sim 1$$

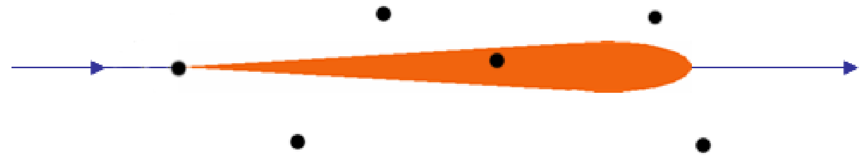


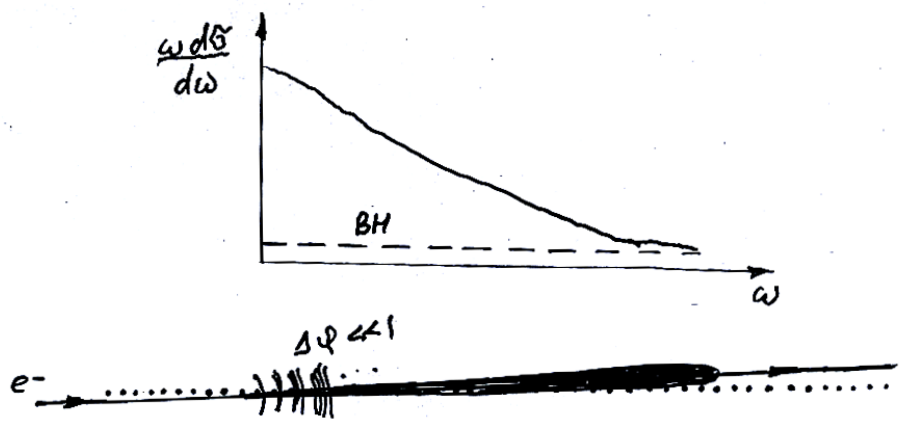
$$\Delta t \sim \frac{2\gamma^2}{\omega}\frac{1}{1 + \gamma^2\overline{\theta^2_{\Delta t}} + \gamma^2\theta^2}$$

$$\Delta t \sim \begin{cases} 2\gamma^2/\omega & \gamma^2\overline{\theta^2_{\Delta t}} << 1 \\ << 2\gamma^2/\omega & \gamma^2\overline{\theta^2_{\Delta t}} >> 1 \end{cases}$$

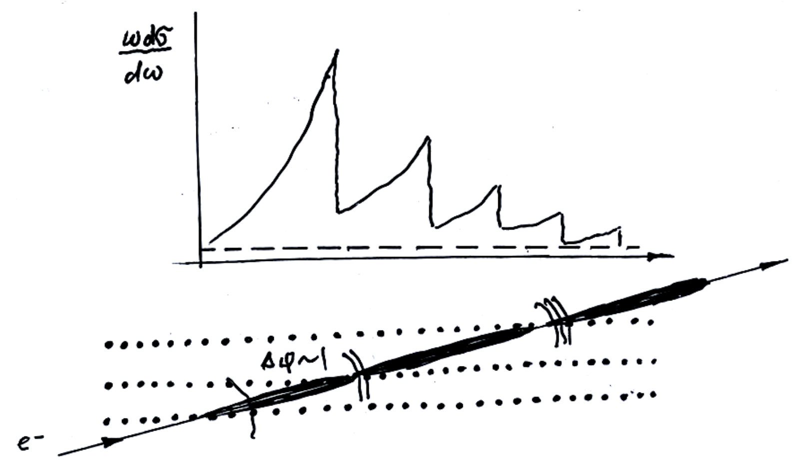
Coherent length

(Ter-Mikaelian 1953 – crystal, Landau-Pomeranchuk 1953 – amorphous media)





Coherent effect



Coherence + Interference

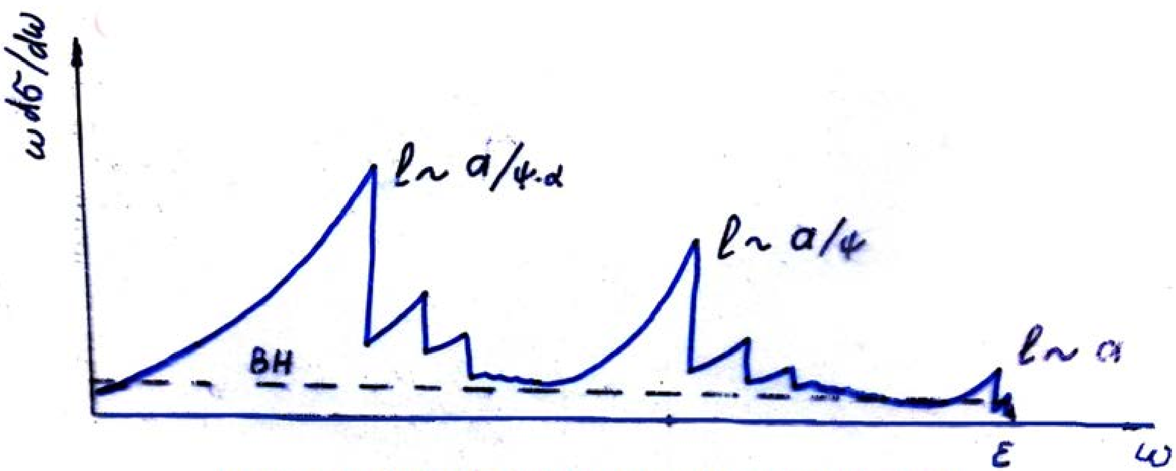
Landau was agreed that Ter-Mikaelian's results were correct, but he said that it is needed to use another way for describing this effect

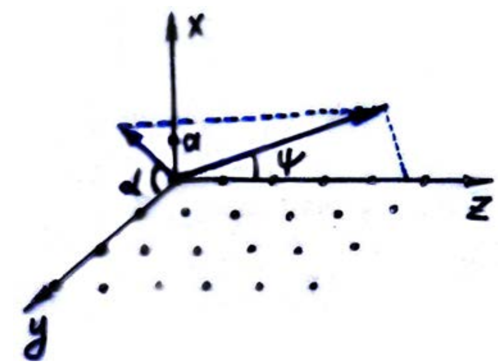
Coherent Bremsstrahlung in Born Approximation

Ferretti 1950, Ter-Mikaelian 1952, Überall 1960

$$\omega\frac{d\sigma}{d\omega}=\frac{2e^2\delta}{m^2\Delta}\frac{\varepsilon'}{\varepsilon}\sum_{\vec{g}}\frac{g_\perp^2}{g_\parallel^2}\left[1+\frac{\omega^2}{2\varepsilon\varepsilon'}-2\frac{\delta}{g_\parallel}\left(1-\frac{\delta}{g_\parallel}\right)\right]\left|U_g\right|^2 e^{-g^2\overline{u^2}}$$

$$q_\parallel \ge \delta = \omega m^2/2\varepsilon\varepsilon', \qquad g_\parallel = g_z + \psi\left(g_y\cos\alpha + g_x\sin\alpha\right) \ge \delta$$

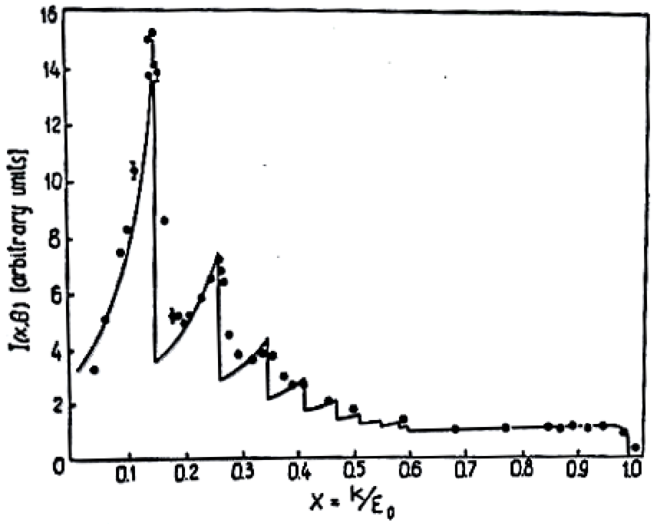


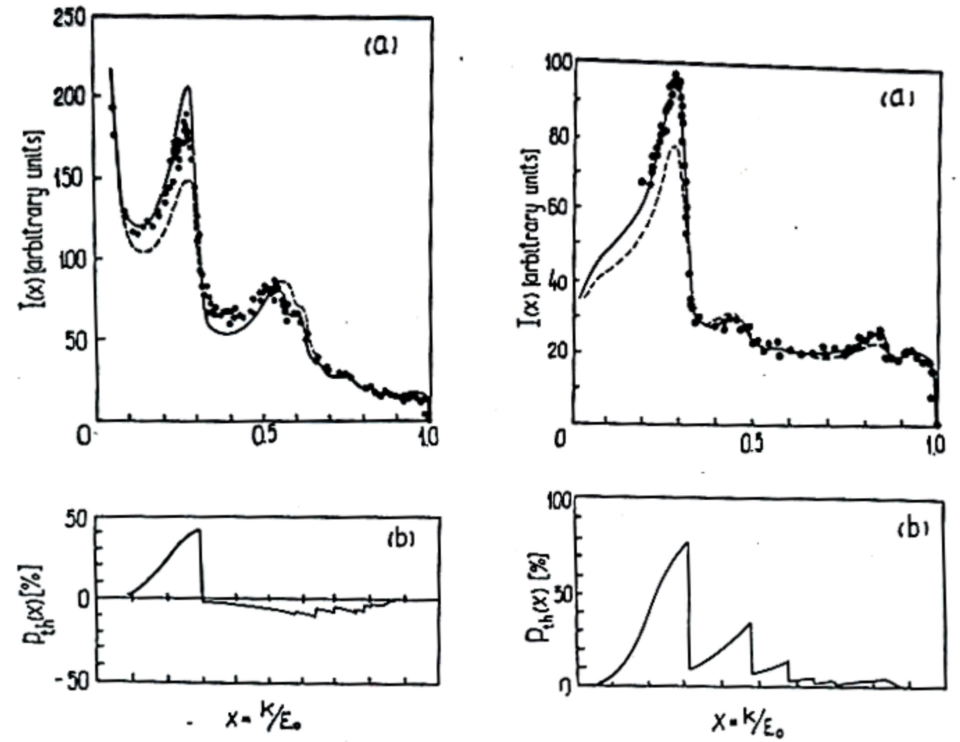


Akhiezer, Shul'ga Sov.Phys.Usp. 1982 v.25 p.54

Experiment $\varepsilon \sim 1-5\ \mathrm{GeV}$ (1962 - 1965)

Frascati, DESY,Kharkov, Protvino, Tomsk, Yerevan, SLAC, …





Frascati

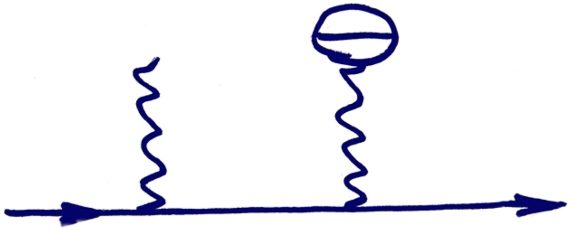
ε=1 GeV, θ =4,6 mrad

DESY

ε=4,8 GeV, θ =3,4 mrad

Generalization of CB theory

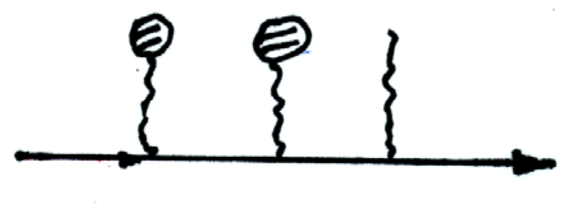
The main idea:

-For  $d\sigma_{coh} >> d\sigma_{atom}$

-The relative contribution of higher Born approximation can be also increased (A.Akhiezer, P.Fomin, N.Shul'ga 1971)

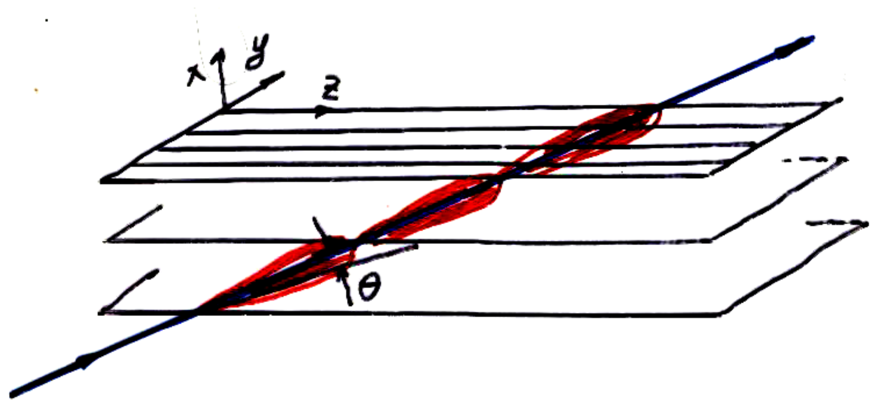
Second Born approximation in CB theory

A.Akhiezer, P.Fomin, N.Shul'ga (1970)



$$d\sigma_c = d\sigma_{coh}^{Born} \cdot \left(1 \pm \eta \frac{\theta_c^{\ 2}}{\theta^2}\right), \qquad \hbar\omega \sim \varepsilon$$

$\eta \sim 1$ $\qquad \theta_c$ – *crytical channelling angle*



Higher Born Approximation in the CB Theory

A.Akhiezer, N.Shul’ga (1975)

$$N_{coh} \sim \min\left(\frac{l_{coh}}{a}, \quad \frac{R}{\psi_a}\right)$$

$$\frac{Ze^2}{\hbar c} \ll 1 \quad \rightarrow \quad N_{coh}\frac{Ze^2}{\hbar c} \sim \frac{R}{\psi a}\frac{Ze^2}{\hbar c} \ll 1$$

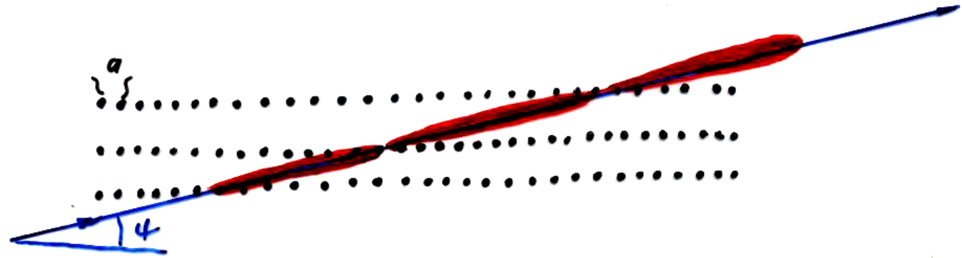
Quickly destroys for $\psi \rightarrow 0$

PARADOX

This condition did not fulfill practically for experiments (1960-1970) on verification of F – T – Ü theoretical results.

But the experiments were in good agreement with this theory !!!
Why ???

Eikonal, Semiclassical, Classical CB Theory



Semiclassical approximation

$$\frac{N_c Ze^2}{\hbar c} = \frac{R}{\psi a}\frac{Ze^2}{\hbar c} \gg 1$$

!!!

Classical Electrodynamics

$$N_c \frac{Ze^2}{\hbar c} \gg 1, \quad \hbar\omega \ll \varepsilon$$

$$d\sigma^{(WKB)} = d\sigma\left\{\vec{r}_{cl}\left(t\right)\right\}$$

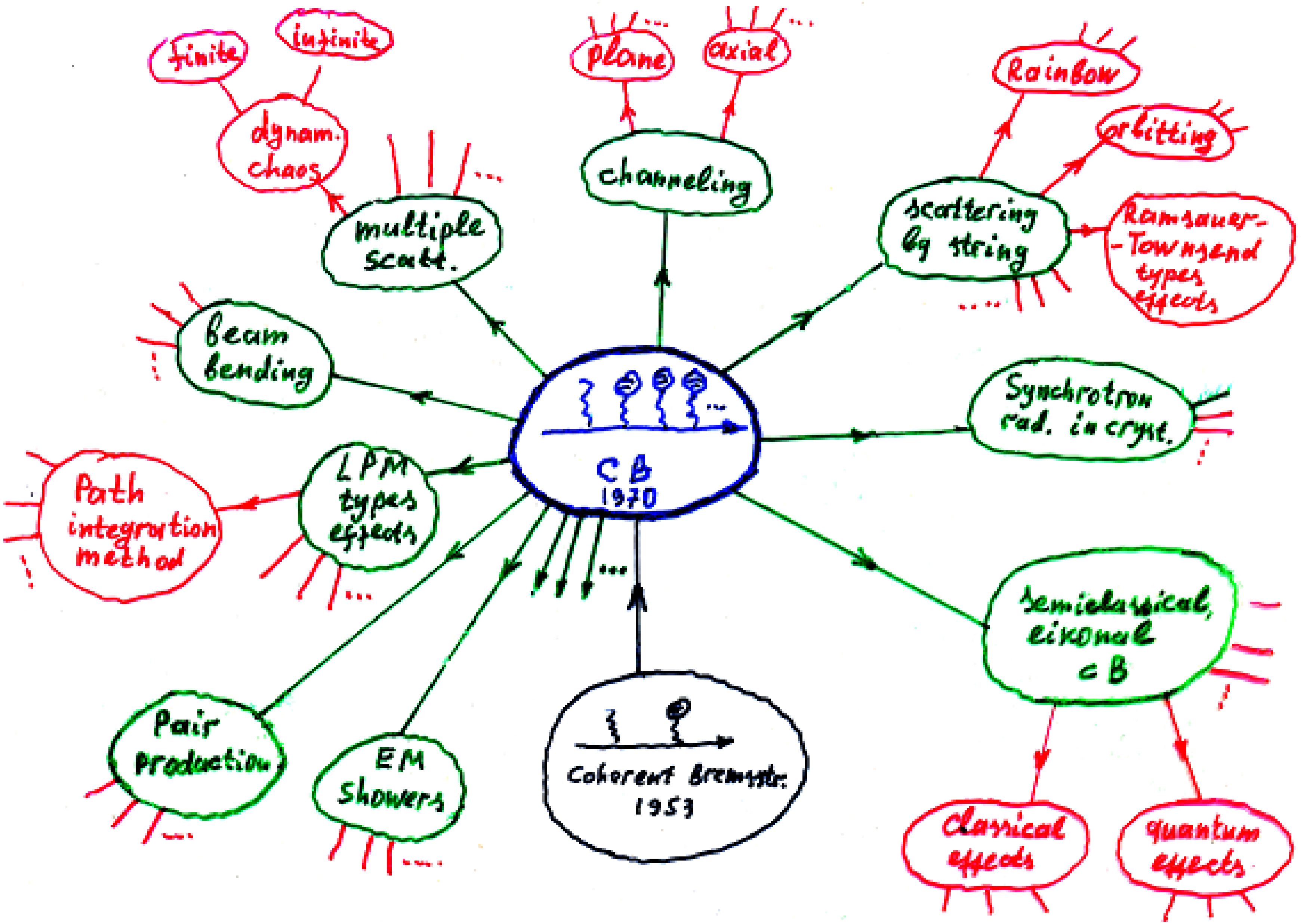
- Radiation is determined by the classical trajectory !!!
- It is necessary to know the types of particles' motion in crystal
- Same methods for description of CB and LPM effects !!!

New area of research

The interaction of high-energy particles with matter in conditions of effectively strong interaction of the particle with atoms of media (semiclassical, classical approximations)

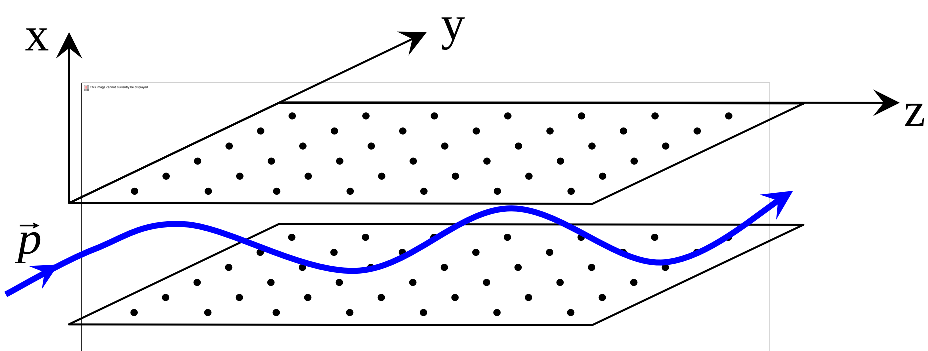
$$N_c \frac{Ze^2}{\hbar c} >> 1$$

Problems generated by the theory of coherent radiation in crystals



Planar Channeling

Lindhard (1965)

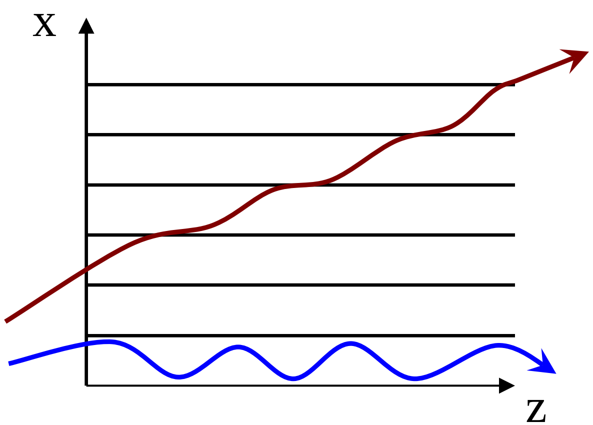


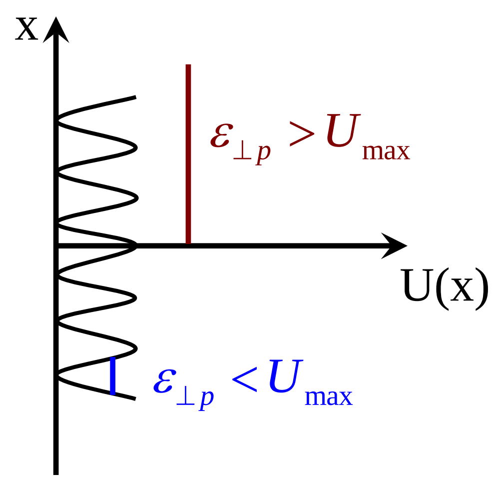
$$U_p(x) = \frac{1}{L_y L_z}\int dydz \sum_n u\left(\vec{r} - \vec{r}_n\right)$$

$p_z = const \approx p$

$$\ddot{x} = -\frac{1}{E}\frac{\partial}{\partial x}U_p(x)$$

$$\varepsilon_{\perp p} = \frac{E\,\dot{x}^2}{2} + U_p(x)$$





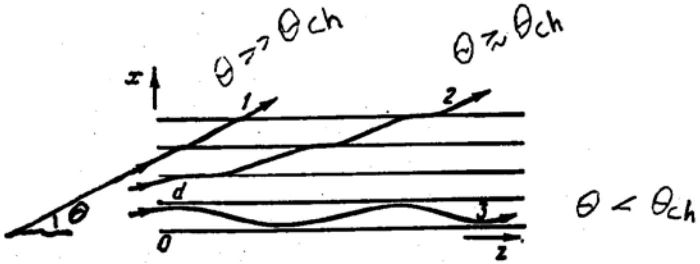
$$\varepsilon_{\perp p} = U_{\max} = \frac{E\,\theta_p^{\,2}}{2}$$

$$\theta_p = \sqrt{\frac{2U_{\max}}{E}}$$

- Above-barrier motion (Akhiezer, Shul'ga 1978)

Coherent and Channeling Radiation

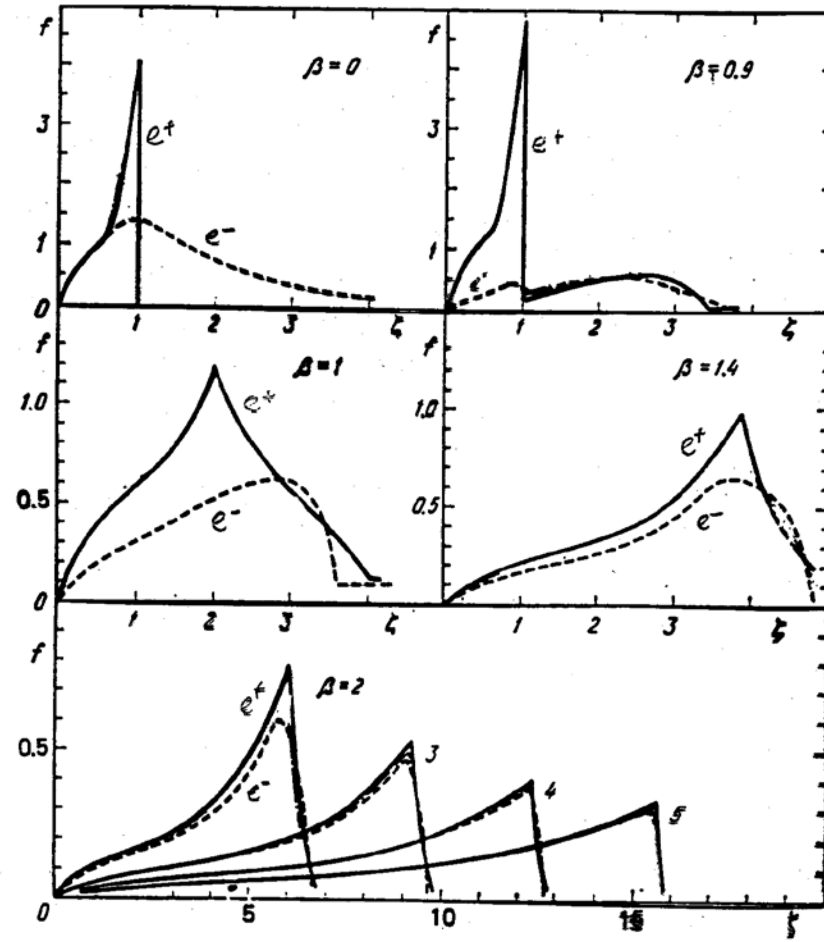
A. Akhiezer, N. Shul'ga (1978)



$\theta \gg \theta_{ch}$ $\qquad$ $\theta \lesssim \theta_{ch}$

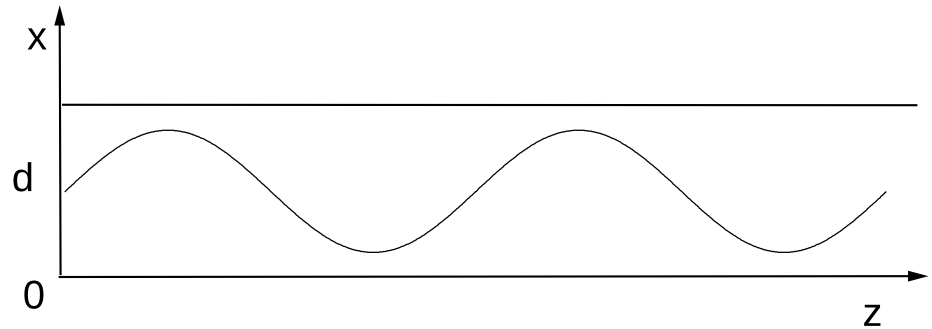
$I'_c \sim \frac{R^2}{\theta a^2} I'_{BH}$ $\qquad$ $I'_c \sim \frac{R^2}{\theta_{ch} a^2} I'_{BH}$

$\gamma \vartheta_{sc} \ll 1$ $\qquad$ $\theta_{ch} = \sqrt{\frac{2U_0}{\varepsilon}}$



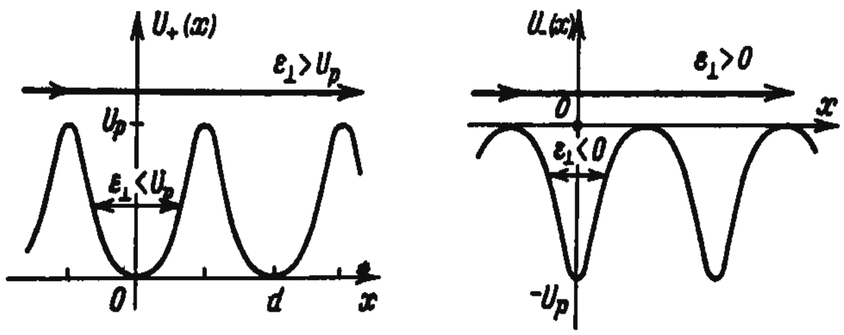
$\beta = \frac{\theta}{\theta_{ch}}$

Radiation at planar channeling

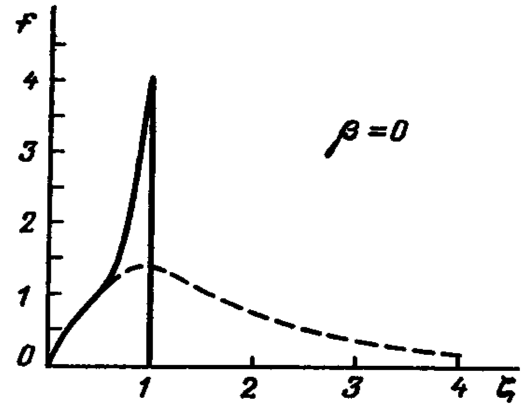


$$U_p(x) = \frac{1}{L_y L_z}\int dy dz \sum_n u(\vec{r} - \vec{r}_n)$$

$$\ddot{x} = -\frac{e}{\varepsilon}\frac{\partial U_p(x)}{\partial x}$$

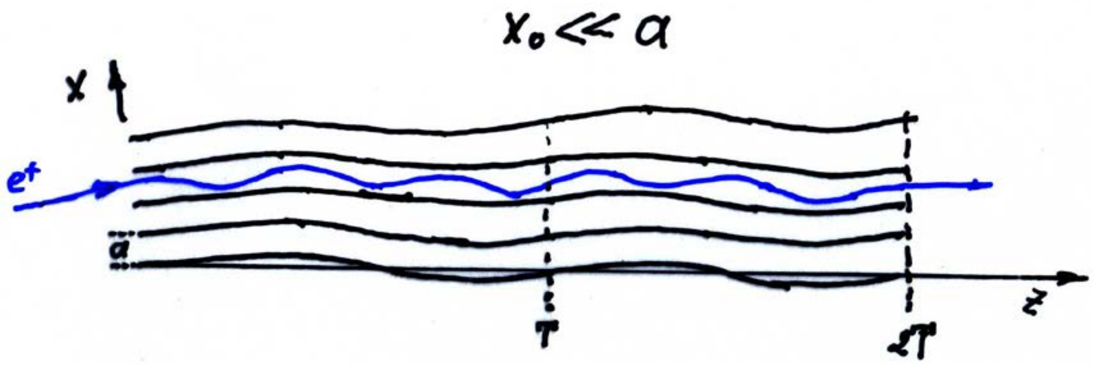


a δ



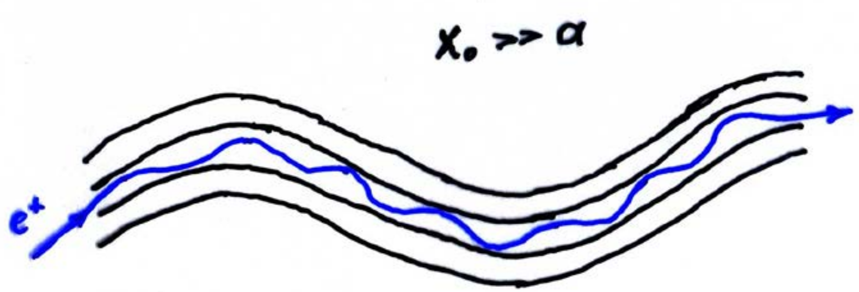
$$\omega_{\max} = 2\gamma^2\Omega_{osc} \sim \varepsilon^{3/2}$$

Channeling radiation in periodically deformed crystal plane



$$U(x,y,z) = U_p(x - x_0 \sin \Omega z),$$

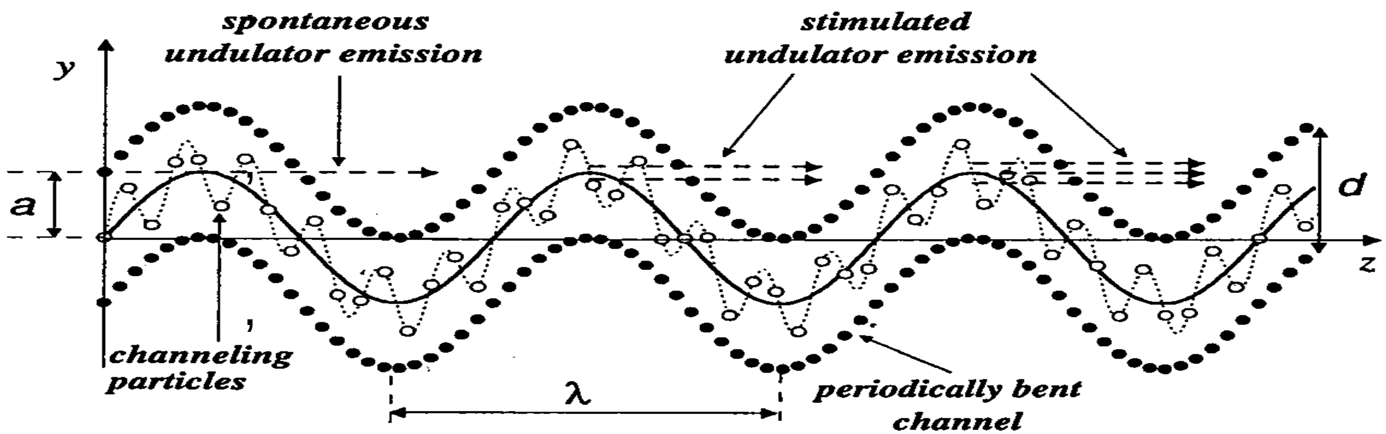
$$\Omega = 2\pi/T.$$



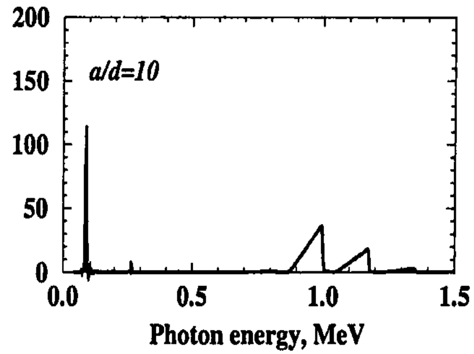
V. Boldyshev (1982)
L. Grigorian, A. Mkrtchyan
et al (2001)

A. Korol, A. Solov'yov,
W. Greiner (2004)
V. Biryukov et al (2006)

Radiation in the field of periodically deformed crystal planes of atoms at canalling



$d << a << \lambda \qquad d \sim 10^{-8} cm \qquad a \sim 10..10^{2} d \qquad a \sim 10^{-5}..10^{-4} \lambda$



1.L.Sh.Grigoryan, A.R.Mkrtchyan, A.H.Mkrtchyan et al, Nucl. Instr. and Meth. in Physics Research B **173**, 132 (2001).

2.A.V.Korol, A.V.Solov`yov and W.Greiner, Int. J. Mod. Phys. E **13**, 867 (2004).

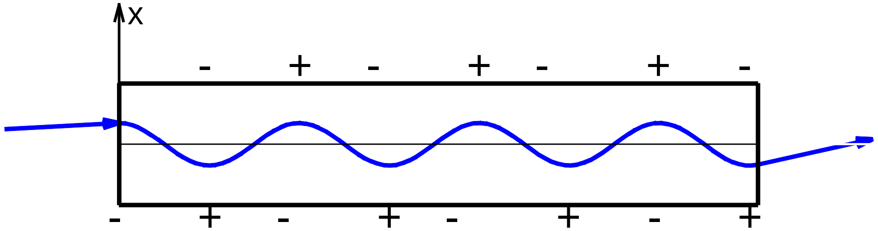
Coherence of undulator radiation for high energy electrons

E. Bulyak, N. Shul'ga

arxiv:1506.03255v2[physics.acc-ph] 11 Sep. 2015

(submitted to Phys. Rev.)

Undulator Radiation



$$\vec{v}(t+T)=\vec{v}(t), \quad \vec{v}_{\parallel} >> \vec{v}_{\perp}$$

$$x(t)=x_m \sin \Omega t \qquad \Omega = 2\pi/T$$

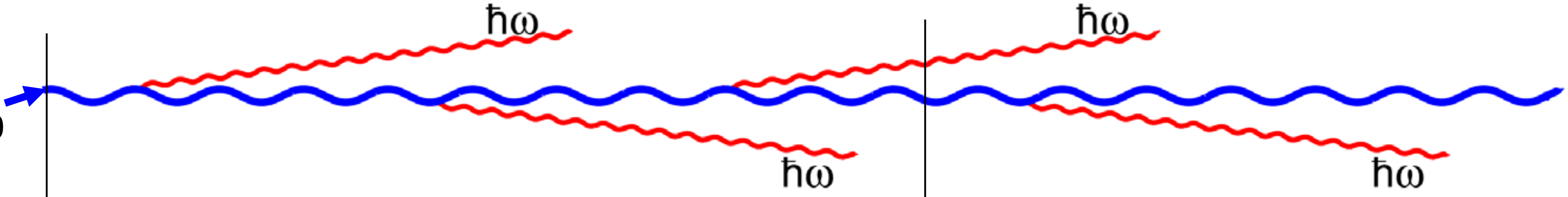
$$z(t)\approx v_{\parallel} t$$

$$\left.\frac{dE}{d\omega}\right|_{\theta=0} \approx \frac{\sin^2\left(N\pi \frac{\omega}{\omega_D}\right)}{\sin^2\left(\pi \frac{\omega}{\omega_D}\right)} \frac{e^2}{\pi^2}\gamma^4 \left| \int_0^T dt \dot{v}_{\perp}(t) e^{i\omega t/2\gamma^2} \right|^2$$

$$\text{for } \omega \approx \omega_D \qquad \omega_D = 2\gamma^2 \Omega$$

$$\left.\frac{dE}{d\omega do}\right|_{\theta=0} \sim N^2$$

The Problem



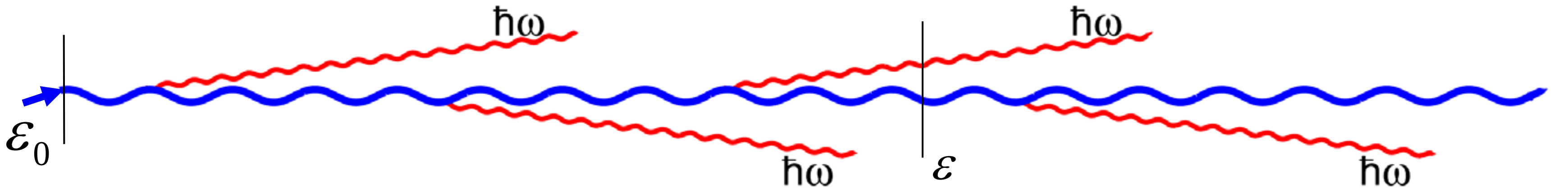
ε_0

$f(0,\varepsilon)=\delta(\varepsilon-\varepsilon_0)$ $f(z,\varepsilon)$

$$f(0,\varepsilon)=\delta(\varepsilon-\varepsilon_0), \qquad \int_0^\infty d\varepsilon\, f(z,\varepsilon)=1$$

When the dependence $\left.\frac{dE}{d\omega do}\right|_{\theta=0} \sim N^2$ is destroyed?

The Kinetic Equation Method



$$\frac{d}{dz} f(z,\varepsilon) = \int_0^\infty d\omega\, w(\varepsilon,\omega)\left[f(z,\varepsilon+\hbar\omega) - f(z,\varepsilon)\right]$$

$$f(z,\varepsilon) = \frac{1}{2\pi}\int dp \exp\left\{ ip(\varepsilon-\varepsilon_0) - z\int_0^\infty d\omega\, w(\varepsilon,\omega)\left[1-e^{ip\omega}\right]\right\}$$

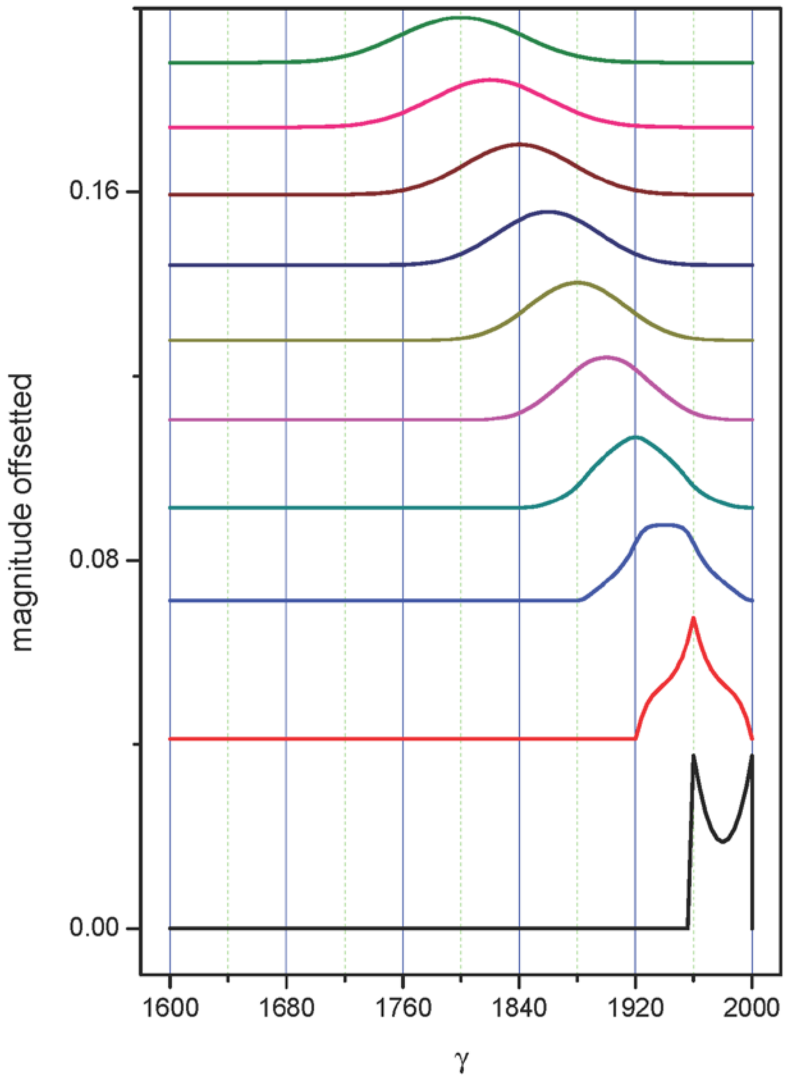
Diffusion approximation $\hbar\omega << \varepsilon$

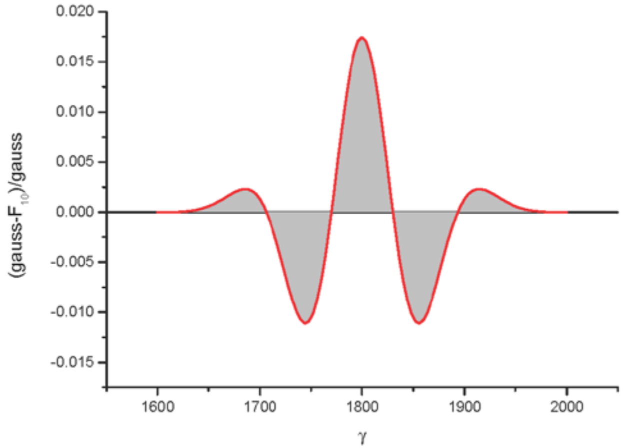
$$f(z,\varepsilon) \approx \frac{1}{\sqrt{2\pi z\overline{\omega^2}}} \exp\left\{ -\frac{(\varepsilon_0-\varepsilon-z\overline{\omega})^2}{2z\overline{\omega^2}}\right\}$$

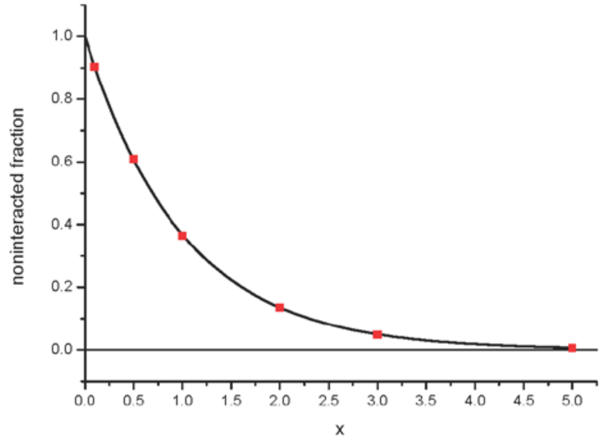
$$\overline{\omega} = \int_0^\infty d\omega\; \omega\; w(\varepsilon,\omega), \qquad \overline{\omega^2} = \int_0^\infty d\omega\, \omega^2 w(\varepsilon,\omega).$$

Specific spectra, deviation from Gaussian

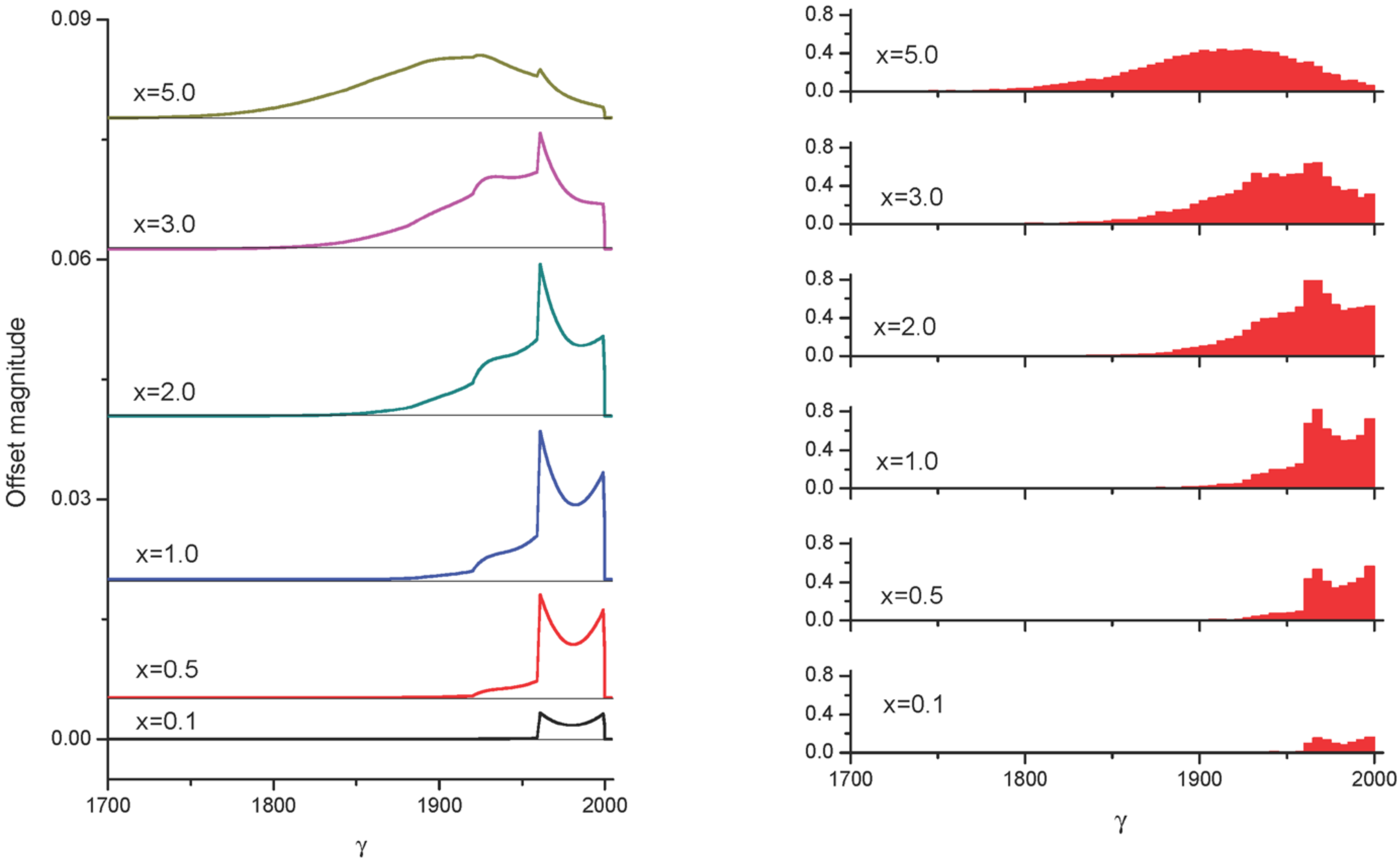
Numbers: Alternative (Compton) positron source for ILC, $\gamma_0 = 2 \times 10^3$; $\omega_{\max} = 40$







Aggregate spectra. Theory ⇔ simulation
Non-interacted not shown



Poisson Distribution

$$f\left(z,\varepsilon\right)=\frac{1}{2\pi}\int dp\exp\left\{ip\left(\varepsilon-\varepsilon_0\right)-z\int_0^\infty d\omega\frac{1}{\hbar\omega}\frac{d^2E\left(\varepsilon,\omega\right)}{d\omega dz}\left(1-e^{ip\omega}\right)\right\}$$

$$\omega^*=\omega_{eff}\sim 2\gamma^2\Omega$$

$$=\frac{1}{2\pi}\int dp\exp\left\{ip\left(\varepsilon-\varepsilon_0\right)-\frac{\Delta E}{\hbar\omega^*}\left(1-e^{ip\omega^*}\right)\right\}=$$

$$\xi=\frac{\Delta E}{\hbar\omega^*}$$ - the mean number of photons

$$=\int\frac{dp}{2\pi}\exp\left\{ip\left(\varepsilon-\varepsilon_0\right)-\xi+\xi e^{ip\omega^*}\right\}\approx\int\frac{dp}{2\pi}e^{ip\left(\varepsilon-\varepsilon_0\right)-\xi}\cdot\sum_{n=0}^\infty\frac{\xi^n e^{inp\omega^*}}{n!}$$

$$f\left(z,\varepsilon\right)=\sum_{n=0}^\infty f_n\left(\xi\right)\delta\left(\varepsilon+n\hbar\omega^*-\varepsilon_0\right)$$

$$f_n\left(\xi\right)\left(z,\varepsilon\right)=\frac{1}{n!}\xi^n e^{-\xi}$$

Coherent effect in undulator radiation

$$\left\langle \left.\frac{dE}{d\omega\, do}\right|_{\vartheta=0} \right\rangle \approx \frac{e^2}{\pi^2}\gamma^4 \int_0^L dt \int_0^L dt' \,\dot{\vec{v}}_\perp(t)\,\dot{\vec{v}}_\perp(t')\, e^{i\frac{\omega}{2\gamma^2}(t-t')} \cdot e^{-\frac{L\overline{\omega^2}}{2}\left(\frac{\omega(t-t')}{m\gamma^3}\right)^2}$$

$$\frac{L\overline{\omega^2}}{2}\left(\frac{\omega L}{m\gamma^3}\right)^2 << 1$$

$$\left\langle \left.\frac{dE}{d\omega\, do}\right|_{\vartheta=0} \right\rangle \approx \frac{\sin^2\left(N\pi\frac{\omega}{\omega_D}\right)}{\sin^2\left(\pi\frac{\omega}{\omega_D}\right)} \frac{e^2}{\pi^2}\gamma^4 \left| \int_0^L dt\, \dot{\vec{v}}_\perp(t)\, e^{i\omega t/2\gamma^2} \right|^2$$

$$\left\langle \left.\frac{dE}{d\omega\, do}\right|_{\vartheta=0} \right\rangle \sim N^2 \quad \text{for} \quad \omega \approx \omega_D$$

Destruction of coherent effect in UR

$$\frac{L\overline{\omega^2}}{2}\left(\frac{\omega L}{m\gamma^3}\right)^2 \gtrsim 1$$

$$w(\varepsilon_0,\omega)=\frac{1}{\hbar\omega}\frac{e^2\pi\kappa^2}{T}\zeta\left(1-2\zeta+2\zeta^2\right)\Theta(1-\zeta)$$

where $\zeta=\omega/\omega_D$, $\kappa=\gamma\vartheta_{\max}$

$$\sqrt{\frac{7e^2\pi\kappa^2}{60}}N^{3/2}\frac{16\pi^2\gamma}{mT}\sim 1$$

For $\kappa=1,\quad T=1\mathrm{cm},\quad \gamma=10^6$

$$N\approx 240$$

THANK YOU FOR YOUR ATTENTION!

Practical application 2: Limitations of coherency
Rough estimation

- Energy spread in electron bunches

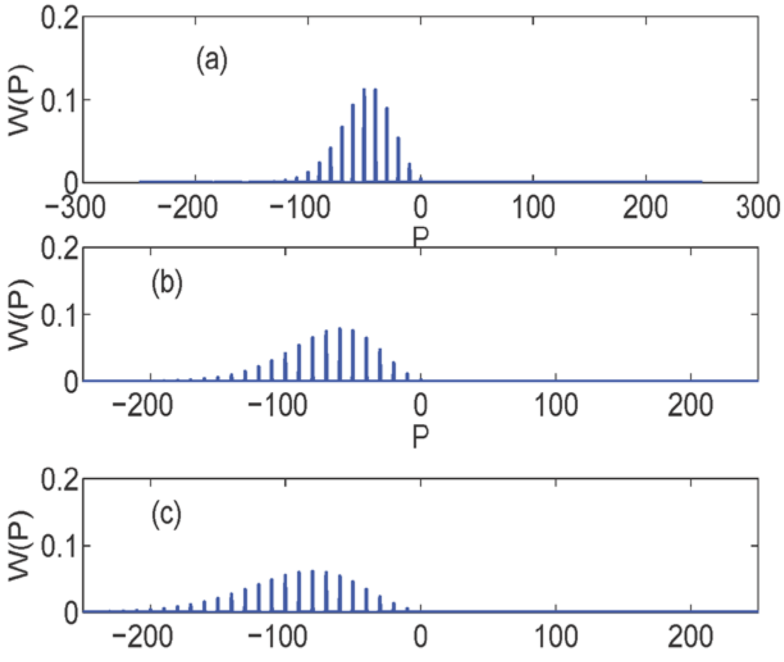
$$\frac{\Delta\gamma}{\gamma} \approx \frac{\sqrt{x\overline{\omega^2}}}{\gamma} \le \frac{1}{2N_{\mathrm{undul}}}$$

- Max number of 'coherent' photons

$$x_{\max} \lesssim \left[\frac{\alpha K^2}{4\pi\sqrt{7/20}}\,\frac{\lambda_{\mathrm{undul}}}{\gamma\lambda_{\mathrm{Comp}}}\right]^{2/3}$$

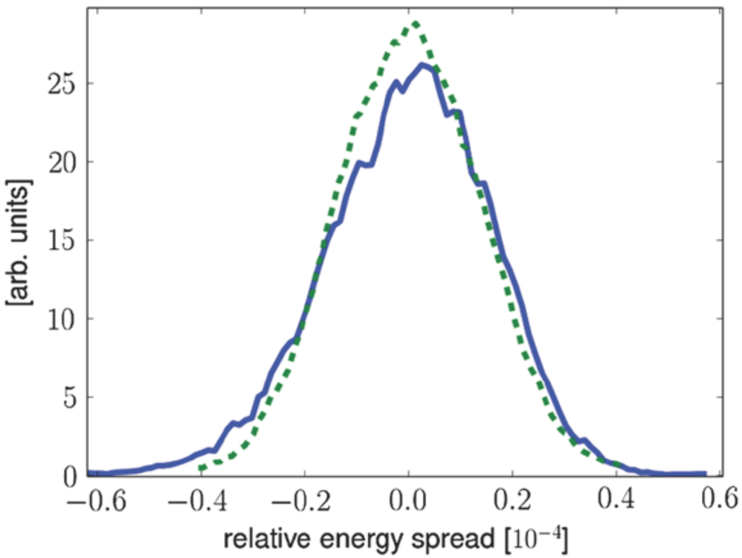
- Numbers:
  - ILC: $\gamma = 3\times 10^5,\ K = 0.45,\ \lambda_{\mathrm{undul}} = 1.1\,\mathrm{cm}$
    max coherent $x_{\max} \approx 600\,\text{periods} \approx 7\,\mathrm{m}$
  - Example: $\gamma = 3\times 10^4,\ K = 2,\ \lambda_{\mathrm{undul}} = 2\,\mathrm{cm}$
    max coherent $x_{\max} \approx 1700\,\text{periods} \approx 34\,\mathrm{m}$

Robb, Bonifacio *Europhysics Letters* 94(3): 34002, 2011



criticised by: Geloni, Kocharyan, Saldin *arXiv:1202.0961* 2012

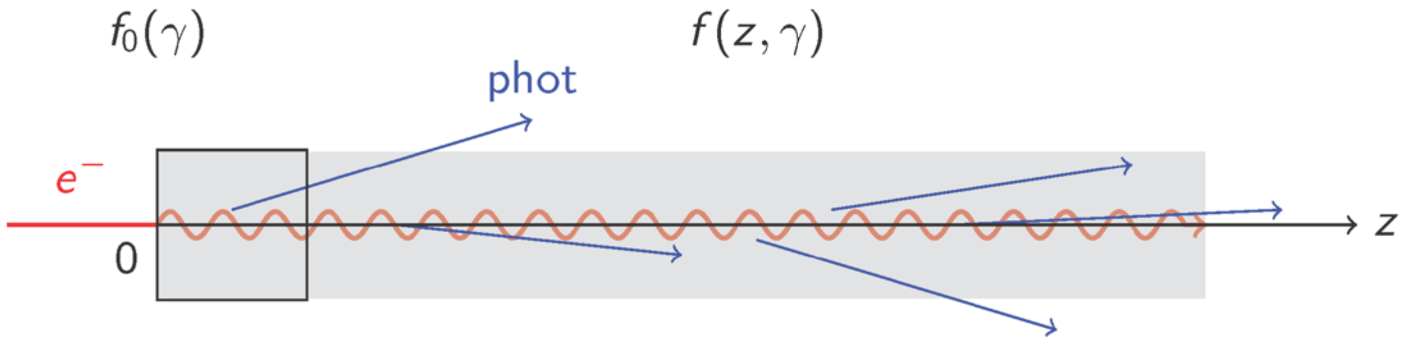
Agapov, Geloni *Phys.Rev.STAB* 17:110704, 2014



Although leading to correct beam energy spread, the diffusion approximation can lead to unphysical single trajectories in the longitudinal phase space, since the electron energy can grow . . . as a result of photon emission

Existing models explore many–quanta regime, $x \gg 1$

Problem setup: ultra–relativistic electrons, quantum recoils



problem setup

- periodic force with given envelop – undulator, laser pulse
- radiation: statistically independent photons, given spectra
- recoils decrease electrons' energy
- goal:evolution of initially given electron spectrum along the field special attention – front end of the field

motivation

- ILC positron source: effect of the undulator on the beam parameters
- ILC alternative Compton positron source – performance
- FEL energy limitations
- lack of theory corresponding to $x \lesssim 1$, drawbacks of the diffusive approximation

Kinetic equation for electron spectrum $f(z,\gamma)$
Landau 1944; Akhiezer, Shul'ga 1996; Khokonov 2004

System of units: $m_e = c = \hbar = 1$, initial distribution $f_0(\gamma) \equiv f(z=0,\gamma)$

$$\boxed{\frac{\partial}{\partial z} f(z,\gamma) = \int \left[f(z,\gamma+\omega) W(z,\gamma+\omega,\omega) - f(z,\gamma) W(z,\gamma,\omega) \right] \mathrm{d}\omega}$$

where $W(z,\gamma,\omega)$ is probablity density
Approximation: $\gamma \gg 1$, $\omega_{\mathrm{max}} \ll \gamma$
generalization: probability of recoils presentation – *envelope–carrier* (Khokonov)
$W(z,\gamma,\omega) = \psi(z) w(\gamma,\omega) \approx \psi(z) w(\gamma_0,\omega) = \psi(z) w(\omega)$, $\int w(\omega)\,\mathrm{d}\omega = 1$
Kinetic equation casts into

$$\boxed{f'_x = f \star w - f}$$

with $f \star w \equiv \int f(\gamma+\omega) w(\omega) d\omega$ cross correlation; $x = \int_0^z \psi(z')\,\mathrm{d}z'$ number of emitted photons.

Kinetic equation: solution and moments

$\boxed{f'_x = f \star w - f}$ Fourier transform: $\hat{f}'_x = (\check{w} - 1)\hat{f} \Rightarrow \hat{f} = \mathrm{e}^{-x}\mathrm{e}^{x\check{w}}\hat{f}_0$

Taylor series for $\mathrm{e}^{x\check{w}}$ and inverse transform yields:

$$f(x,\gamma) = \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-x}x^n}{n!} F_n(\gamma) = \{\mathrm{P}(x) \cdot \mathrm{F}(\gamma)\} ,$$

where $F_0(\gamma) = f_0(\gamma) = f(x=0,\gamma)$ is an initial distribution (spectrum),

$$F_n(\gamma) = (w \star w \star \cdots \star w \star f_0)(\gamma) = (w \star F_{n-1})(\gamma) , \qquad n = 1, 2 \ldots$$

in traditional form

$$F_n(\gamma) = \int F_{n-1}(\gamma + \omega) w(\omega)\, \mathrm{d}\omega$$

Mean energy (1st moment) and variance (2nd moment)

$$\overline{\gamma}(x) = \overline{\gamma_0} - x\,\overline{\omega} ; \qquad \mathrm{Var}[\gamma](x) = \overline{(\gamma - \overline{\gamma})^2} = \mathrm{Var}[\gamma_0] + x\,\overline{\omega^2}$$

Relation with distance along axis: $z \rightarrow x(z) = \int_0^z \psi(z')\,\mathrm{d}z'$.

Initial delta–spectrum of electrons, $f_0 = \delta(\gamma - \gamma_0)$
Spectrum of recoils

$$w(\omega) = \frac{3}{2\omega_{\max}}\left[1 - \frac{2\omega}{\omega_{\max}}\left(1 - \frac{\omega}{\omega_{\max}}\right)\right]\Pi\left(\frac{\omega}{\omega_{\max}} - \frac{1}{2}\right)$$

Rect function $\Pi(y) = 1$ if $|y| < 1/2$ and $= 0$ beyond
Moments:

$$\overline{\gamma}(x) = \gamma_0 - x\,\omega_{\max}/2; \quad \mathrm{Var}[\gamma](x) = \frac{7}{20}x\omega_{\max}^2$$

Exact solution

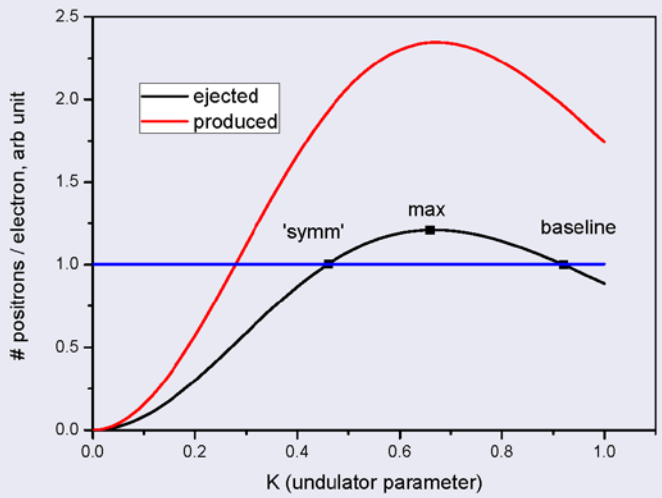
$$F_n(\gamma) = \int \mathrm{e}^{2\pi\mathrm{i}s(\gamma - \gamma_0 + n\overline{\omega})}\left[\frac{3}{16}\,\frac{(4\pi^2 s^2\overline{\omega}^2 - 1)\sin(2\pi s\overline{\omega}) + 2\pi s\overline{\omega}\cos(2\pi s\overline{\omega})}{(\pi s\overline{\omega})^3}\right]^n \mathrm{d}s\,,$$

where $\overline{\omega} = \omega_{\max}/2$

LINEAR COLLIDER COLLABORATION

Designing the world's next great particle accelerator

yield of positrons



param	base	max	'symm'
K	0.92	0.66	0.46
N_{phot}	330	170	83
$N_{phot}^{1\ harm}$	141	110	67
E_γ^{max}, MeV	10.3	13.2	15.6
ΔE_e, GeV	-3.3	-1.7	-0.83
σ_e/γ, %	0.14	0.1	0.07

'symm' vs. baseline (the same yield)

- 4 times less power load upon collimators and a convertor
- smaller phase volume of produced positrons (expected)