Estimates for distributions of Hölder semi-norms of random processes from spaces $\mathbb{F}_\psi(\Omega)$

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We provide estimates for distributions of semi-norms of sample functions of random processes from spaces $\mathbb{F}_\psi(\Omega)$, defined on a compact space and on an infinite interval $[0, \infty)$, in Hölder spaces.

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1 Introduction

$\mathbb{F}_\psi(\Omega)$ spaces of random variables and processes belonging to these spaces are investigated by Kozachenko and Mlavets' [5].

In the following we deal with estimates of distributions of Hölder semi-norms of sample functions of random processes from spaces $\mathbb{F}_\psi(\Omega)$, i.e. probabilities

$$\mathsf{P}\left\{\sup_{\substack{0<\rho(t,s)\le\varepsilon\\ t,s\in\mathbb{T}}} \frac{|X(t)-X(s)|}{f(\rho(t,s))} > x\right\}.$$

Such estimates and assumptions under which semi-norms of sample functions of random processes from spaces $\mathbb{F}_\psi(\Omega)$, defined on a compact space, satisfy the Hölder condition were obtained by Zatula and Kozachenko [7]. Similar results were provided for Gaussian processes, defined on a compact space, by Dudley [3]. Buldygin and Kozachenko [2] generalized Dudley's results for random processes belonging to Orlicz spaces. Marcus and Rosen [4] obtained L^p moduli of continuity for a wide class of continuous Gaussian processes. Kozachenko et al. [6] studied the Lipschitz continuity of generalized sub-Gaussian processes and provided estimates for the distribution of Lipschitz norms of such processes. But all these problems were not considered yet for processes, defined on an infinite interval.

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2 Preliminaries

Definition 1. Let $\psi(u) > 0$, $u \geq 1$ be some increasing function such that $\psi(u) \to \infty$ as $u \to \infty$. We say that a random variable ξ belongs to the space $\mathbb{F}_\psi(\Omega)$ (see [5]) if

$$\sup_{u\geq 1} \frac{(\mathsf{E}|\xi|^u)^{1/u}}{\psi(u)} \leq \infty.$$

It is proved in the paper [5] that $\mathbb{F}_\psi(\Omega)$ is a Banach space with respect to the norm

$$\|\xi\|_\psi = \sup_{u\geq 1} \frac{(\mathsf{E}|\xi|^u)^{1/u}}{\psi(u)}.$$

Theorem 1 ([5])**.** *If a random variable ξ belongs to the space $\mathbb{F}_\psi(\Omega)$, then*

$$\mathsf{P}\{|\xi| > x\} \leq \inf_{u\geq 1} \frac{\|\xi\|_\psi^u \cdot (\psi(u))^u}{x^u}$$

for all $x > 0$.

Let ξ_1, ..., ξ_n be random variables belonging to the space $\mathbb{F}_\psi(\Omega)$. Put $\eta_n = \max_{1\leq k\leq n} |\xi_k|$, $a_n = \max_{1\leq k\leq n} \|\xi_k\|_\psi$.

Definition 2. An $\mathbb{F}_\psi(\Omega)$ space has the property Z if there are monotonically non-decreasing function $z(x) > 0$, monotonically increasing function $U(n)$ and the real number $x_0 > 0$ such that for all sequence of random variables $(\xi_k,\, k = \overline{1,n})$ from the space $\mathbb{F}_\psi(\Omega)$, $\forall x > x_0$ and for all $n \geq 2$ the following holds

$$\mathsf{P}\{\eta_n > x \cdot a_n \cdot U(n)\} \leq \frac{1}{n} \exp\{-z(x)\}.$$

Definition 3 ([5])**.** We say that a random process $X = \{X(t),\, t \in \mathbb{T}\}$ belongs to the space $\mathbb{F}_\psi(\Omega)$ if random variables $X(t)$ belong to $\mathbb{F}_\psi(\Omega)$ for all $t \in \mathbb{T}$.

Definition 4 ([2])**.** Let $(\mathbb{T},\, \rho)$ be a metric space. The metric massiveness $N(u) := N_{(\mathbb{T},\,\rho)}(u)$ is the minimal number of closed balls (defined with respect to the metric ρ) that cover $\mathbb{T}$ and that have radiuses which do not exceed u.

Definition 5 ([2])**.** A function $q = \{q(t), t \in \mathbb{R}\}$ is called the modulus of continuity if $q(t) \geq 0$, $q(0) = 0$ and $q(t+s) \leq q(t) + q(s)$ for $t > 0$ and $s > 0$.

Definition 6 ([1])**.** A function $v(x)$ satisfy Hölder condition with exponent $\alpha \in (0, 1]$ if the following value is finite:

$$[v]_{\alpha,\mathbb{T}} = \sup_{\substack{t,s\in\mathbb{T}\\ t\neq s}} \frac{|v(t) - v(s)|}{|t-s|^\alpha}.$$

Hölder space $C^{0,\alpha}(\overline{\mathbb{T}})$ is a space of all continuous functions such that the Hölder condition is satisfied with exponent α in the space $\mathbb{T}$.

In the present we deal with a generalization of the semi-norm $[v]_{\alpha,\mathbb{T}}$ in the space $C^{0,\alpha}(\overline{\mathbb{T}})$. Let's consider the quantity

$$[v]_{q,\rho,\mathbb{T}} = \sup_{\substack{t,s\in\mathbb{T}\\ t\neq s}} \frac{|v(t)-v(s)|}{q(\rho(t,s))},$$

where ρ is a metric in the space $\mathbb{T}$, and $q = \{q(t), t \in \mathbb{T}\}$ is a modulus of continuity such that $\exists\, \alpha \in (0,1]\ \forall t,s \in \mathbb{T}, t \neq s : q(\rho(t,s)) \leq |t-s|^{\alpha}$.

3 Main results

In this section we formulate theorems on estimates for distributions of Hölder semi-norms and moduli of continuity of random processes from spaces $\mathbb{F}_{\psi}(\Omega)$, defined on a compact space and on infinite interval.

Theorem 2. *Let $(\mathbb{T}, \rho)$ be a metric compact space. Consider a separable random process $X = \{X(t), t \in \mathbb{T}\}$ belonging to the space $\mathbb{F}_{\psi}(\Omega)$ that has the property Z with functions $U(n)$, $z(x)$ and $x_0 > 0$.*

Suppose that there is a monotonically increasing continuous function $\sigma = \{\sigma(h), h \geq 0\}$ such that $\sigma(h) > 0$ as $h > 0$, $\sigma(0) = 0$ and the following inequality holds

$$\sup_{\rho(t,s)\leq h} \|X(t)-X(s)\|_{\psi} \leq \sigma(h). \tag{1}$$

Let $N(\varepsilon) = N_{\rho}(\mathbb{T}, \varepsilon)$ be a metric massiveness of the space $(\mathbb{T}, \rho)$. Consider $\varepsilon_0 = \sigma^{(-1)}\left(\sup_{t,s\in\mathbb{T}} \rho(t,s)\right)$, where $\sigma^{(-1)}(h)$ is the inverse function of the function $\sigma(h)$, and

$$g_B(\varepsilon) = \int_0^{\sigma(\varepsilon)} U(B^2 N^2(\sigma^{(-1)}(t)))dt < \infty, \qquad \varepsilon > 0.$$

Then for $x > x_0$, $\varepsilon \in (0, \varepsilon_0)$ and $B > 1$ the following inequality holds true

$$\mathsf{P}\left\{ \sup_{0<\rho(t,s)\leq\varepsilon} \frac{|X(t)-X(s)|}{(6+4\sqrt{2})f_B(\rho(t,s)) + (5+2\sqrt{6})g_B(\rho(t,s))} > x \right\} \leq$$
$$\leq \frac{2B(2B+1)}{(B^2-1)N(\varepsilon)} \cdot \exp\{-z(x)\},$$

where $f_B(\varepsilon) = \int\limits_0^{\sigma(\varepsilon)} U(BN(\sigma^{(-1)}(t)))dt,\ \varepsilon > 0.$

Theorem 3. *Let the assumptions of Theorem 2 hold true. Then the following inequality holds*

$$\limsup_{\varepsilon\downarrow 0} \frac{\Delta(X;\varepsilon)}{x_0((6+4\sqrt{2})f_B(\varepsilon)+(5+2\sqrt{6})g_B(\varepsilon))} \le 1$$

with probability 1, where

$$\Delta(X;\varepsilon) = \sup_{\substack{t,s\in\mathbb{T}\\ 0<\rho(t,s)\le\varepsilon}} |X(t)-X(s)|,$$

$$f_B(\varepsilon) = \int\limits_0^{\sigma(\varepsilon)} U(BN(\sigma^{(-1)}(t)))dt,\ g_B(\varepsilon) = \int\limits_0^{\sigma(\varepsilon)} U(B^2N^2(\sigma^{(-1)}(t)))dt < \infty.$$

Now consider an infinite interval $[0,\infty)$. Let $[0,\infty) = \bigcup\limits_{i=0}^{\infty} A_i$, where $A_i = [a_i, a_{i+1}]$ and $\{a_i, i = 0,1,...,\infty\}$ is an increasing sequence, $a_0 = 0$. Denote $\alpha_i = a_{i+1} - a_i$ and $D_i = [a_i, a_{i+1}+\theta]$, where $\theta \in \left(0, \min\limits_{i\ge 0}\alpha_i\right)$. Let $N_i(\varepsilon)$ be metric massiveness for D_i, $i = 0,1,...$ with the metric $\rho(t,s) = |t-s|$, $t,s\in[0,\infty)$.

Theorem 4. *Consider a separable random process* $X = \{X(t), t\in[0,\infty)\}$ *belonging to the Banach space* $\mathbb{F}_\psi(\Omega)$ *that has the property Z with functions* $U(n)$, $z(x)$ *and* $x_0 > 0$. *Suppose that there are monotonically increasing continuous functions* $\sigma_i = \{\sigma_i(h),\ h\ge 0\}$ *such that* $\sigma_i(0) = 0, i = 0,1,...$ *and* $\forall i = 0,1,...$ *the following inequality holds*

$$\sup_{\substack{|t-s|\le h\\ t,s\in D_i}} \|X(t)-X(s)\|_\psi \le \sigma_i(h), \qquad 0 < h < \alpha_i + \theta. \tag{2}$$

Let also

$$\varepsilon_0 = \min_{i\ge 0}\left\{\sigma_i^{(-1)}\left(\sup_{t,s\in D_i}\rho(t,s)\right)\right\} = \min_{i\ge 0}\left\{\sigma_i^{(-1)}(\alpha_i+\theta)\right\},$$

where $\sigma_i^{(-1)}(h)$ *are inverse functions to functions* $\sigma_i(h)$, $i = 0,1,...,$ *and* $\forall i = 0,1,...:$

$$g_{B,i}(\varepsilon) = \int\limits_0^{\sigma_i(\varepsilon)} U(B^2N_i^2(\sigma_i^{(-1)}(t)))dt < \infty;$$

$$f_{B,i}(\varepsilon) = \int\limits_0^{\sigma_i(\varepsilon)} U(BN_i(\sigma_i^{(-1)}(t)))dt, \ \varepsilon > 0.$$

Denoting $w_{B,i}(t,s) = (6+4\sqrt{2})f_{B,i}(|t-s|) + (5+2\sqrt{6})g_{B,i}(|t-s|)$, $t,s \in D_i$ *and* $w_B(t,s)$ *is such function that*

$$w_B(t,s) = \{w_{B,i}(t,s) \,|\, t,s \in A_i \ \textit{or} \ \min\{t,s\} \in A_i, \, \max\{t,s\} \in A_{i+1}\},$$

we obtain that for all $x > x_0$, $\varepsilon \in (0, \min\{\varepsilon_0, \theta\})$ *and* $\theta > \varepsilon$ *under the condition* $\sum\limits_{i=0}^{\infty} \frac{1}{\alpha_i} < \infty$ *the following inequality holds true:*

$$\mathsf{P}\left\{ \sup_{\substack{0<|t-s|\leq\varepsilon \\ t,s\in[0,\infty)}} \frac{|X(t)-X(s)|}{w_B(t,s)} > x \right\} \leq \frac{4\varepsilon B(2B+1)}{B^2-1} \cdot \exp\{-z(x)\} \cdot \sum_{i=0}^{\infty} \frac{1}{\alpha_i + \varepsilon}.$$

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