Theoretical and simulation results on heavy-tailed fractional Pearson diffusions

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We define heavy-tailed fractional reciprocal gamma and Fisher-Snedecor diffusions by a non-Markovian time change in the corresponding Pearson diffusions. We illustrate known theoretical results regarding these fractional diffusions via simulations.

Keywords: Fractional diffusion, Pearson diffusion, Stable subordinator, Transition density, Simulations

1 Introduction

Every continuous distribution with density satisfying the so called Pearson equation

$$\frac{\mathfrak{p}'(x)}{\mathfrak{p}(x)} = \frac{(a_1 - 2b_2)x + (a_0 - b_1)}{b_2x^2 + b_1x + b_0} \tag{1}$$

is called a Pearson distribution (see [8]). The family of Pearson distributions consists of six parametric subfamilies: normal, gamma, beta, Fisher-Snedecor, reciprocal gamma and Student distributions.

Strong solution of SDE

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t,\ t \geq 0, \tag{2}$$

where

$$\mu(x) = a_0 + a_1x,\ \sigma(x) = \sqrt{2b(x)} = \sqrt{2(b_2x^2 + b_1x + b_0)}$$

is called the Pearson diffusion. They are called after Pearson since their stationary distributions belong to the Pearson family. Usually, it is convenient to re-parametrize drift and squared diffusion:

$$\mu(x) = -\theta(x - \mu),\ \sigma^2(x) = 2\theta k(B_2x^2 + B_1x + B_0),$$

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where $\mu \in \mathbb{R}$ is the stationary mean depending on coefficients of the Pearson equation (1), $\theta > 0$ is the scaling of time determining the speed of the mean reversion, and k is a positive constant. Note that we need $\sigma^2(x) > 0$ on the diffusion state space (l, L).

Pearson diffusions could be categorized into six subfamilies, according to the degree of the polynomial $b(x)$ and, in the quadratic case $b(x) = b_2x^2 + b_1x + b_0$, according to the sign of its leading coefficient b_2 and the sign of its discriminant Δ:

- constant $b(x)$ - Ornstein-Uhlenbeck (OU) process with normal stationary distribution,
- linear $b(x)$ - Cox-Ingersol-Ross (CIR) process with gamma stationary distribution,
- quadratic $b(x)$ with $b_2 < 0$ - Jacobi diffusion with beta stationary distribution,
- quadratic $b(x)$ with $b_2 > 0$ and $\Delta > 0$ - Fisher-Snedecor (FS) diffusion with the Fisher-Snedecor stationary distribution,
- quadratic $b(x)$ with $b_2 > 0$ and $\Delta = 0$ - reciprocal gamma (RG) diffusion with reciprocal gamma stationary distribution,
- quadratic $b(x)$ with $b_2 > 0$ and $\Delta < 0$ - Student diffusion with the Student stationary distribution.

2 Fractional diffusions

The subject of our interest are fractional derivatives of order $0 < \alpha < 1$. We define Caputo fractional derivative of order $0 < \alpha < 1$ as

$$\frac{d^\alpha f(x)}{dx^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty \frac{d}{dx} f(x-y) y^{-\alpha}\, dy,$$

or equivalently for absolutely continuous functions as

$$\frac{d^\alpha f(x)}{dx^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-y)^{-\alpha} f'(y)\, dy.$$

Interesting and detailed read regarding fractional derivatives one can find in [7, Chapter 2].

By $(X(t),\, t \geq 0)$ denote the Pearson diffusion solving (2). Introduce $(D_t,\, t \geq 0)$, the standard stable subordinator with index $0 < \alpha < 1$, which is independent of the process $(X(t),\, t \geq 0)$. D_t is a homogeneous Lèvy process with the Laplace transform

$$\mathbb{E}[e^{-sD_t}] = \exp\{-ts^{\alpha}\}.$$

Its inverse process

$$E_t = \inf\{x > 0 : D_x > t\}$$

is non-Markovian, non-decreasing, and for every t random variable E_t has a density, which will be denoted by $f_t(\cdot)$. The Laplace transform of this density is (see e.g., [9])

$$\mathbb{E}[e^{-sE_t}] = \int_0^{\infty} e^{-sx} f_t(x)dx = \mathcal{E}_{\alpha}(-st^{\alpha}), \tag{3}$$

where

$$\mathcal{E}_{\alpha}(z) := \sum_{j=0}^{\infty} \frac{(z)^j}{\Gamma(1+\alpha j)}$$

is the Mittag-Leffler function (see, for example [10]).
Notice that for $\alpha = 1$

$$\mathcal{E}_{\alpha}(z) = e^{z},$$

i.e, Mittag-Leffler reduces to the exponential function.

Now, define the fractional Pearson diffusion $(X_{\alpha}(t),\, t \geq 0)$ as a composition of the Pearson diffusion and inverse of the stable subordinator, i.e.

$$X_{\alpha}(t) = X(E_t),\ \ t \geq 0. \tag{4}$$

We emphasize that $(X_{\alpha}(t),\, t \geq 0)$ is a non-Markovian process and define its transition density $p_{\alpha}(x,t;y)$ as

$$P(X_{\alpha}(t) \in B | X_{\alpha}(0) = y) = \int_B p_{\alpha}(x,t;y)dx \tag{5}$$

for any Borel subset B of (l, L).
Using results from [1] one can show that if the non-fractional Pearson diffusion satisfy SDE (2) with inital condition $X(0) = 0$, then the corresponding fractional Pearson diffusion defined with (4) satisfy SDE

$$dX_{\alpha}(t) = \mu(X_{\alpha}(t))dE_t + \sigma(X_{\alpha}(t))dB_{E_t} \tag{6}$$

with initial condition $X_\alpha(0) = 0$.
Integral form of this SDE is

$$X_\alpha(t) = X(E_t) = \int_0^{E_t} (a_0 + a_1 X(s))\, ds + \int_0^{E_t} \sqrt{2(b_0 + b_1 X(s) + b_2 (X(s))^2)} dB(s).$$

For details we refer to [1] and [4].
Non-heavy-tailed fractional Pearson diffusions (fractional Ornstein-Uhlenbeck (OU) , Cox-Ingersol-Ross (CIR) and Jacobi diffusion) are studied in detail in [3], while heavy-tailed fractional Pearson diffusions (fractional Fisher-Snedecor and reciprocal gamma diffusion) are studied in the recent paper [4]. Fractional Student diffusion have not yet been studied in detail since the nature behind the process (infinitesimal generator and spectrum) is much more complicated then in the other five cases. However, regarding non-fractional Student diffusion one can find some results in [5].
In this paper, we present simulation results regarding fractional Fisher-Snedecor and fractional reciprocal gamma diffusions, which illustrates theoretical results obtained in [4]. Therefore, we begin by stating the necessary theoretical results.

3 Fractional reciprocal gamma diffusion

The reciprocal gamma diffusion satisfies the SDE

$$dX_t = -\theta \left(X_t - \frac{\gamma}{\beta - 1} \right) dt + \sqrt{\frac{2\theta}{\beta - 1} X_t^2}\, dW_t, \quad t \geq 0,$$

with $\theta > 0$ and has invariant density

$$\mathfrak{rg}(x) = \frac{\gamma^\beta}{\Gamma(\beta)}\, x^{-\beta-1} e^{-\frac{\gamma}{x}} \mathrm{I}_{\langle 0,\infty \rangle}(x) \tag{7}$$

with parameters $\gamma > 0$ and $\beta > 1$, where the latter requirement ensures the existence of the stationary mean $\gamma/(\beta - 1)$.

Theorem 1. *The transition density of the fractional RG diffusion is given by*

$$\begin{aligned} p_\alpha(x, t; x_0) = &\sum_{n=0}^{\lfloor \frac{\beta}{2} \rfloor} \mathfrak{rg}(x)\, B_n(x)\, B_n(x_0)\, \mathcal{E}_\alpha(-\lambda_n t^\alpha) \\ &+ \frac{\mathfrak{rg}(x)}{4\pi} \int_{\frac{\theta\beta^2}{4(\beta-1)}}^{\infty} \mathcal{E}_\alpha(-\lambda t^\alpha)\, b(\lambda)\, \psi(x, -\lambda)\, \psi(x_0, -\lambda)\, d\lambda, \end{aligned} \tag{8}$$

where B_n are normalized Bessel polynomials, λ_n are their eigenvalues, $b(\lambda)$ is a constant depending on λ and ψ is the solution of the corresponding Sturm-Liouville equation.

For proof and details see [4].

4 Fractional Fisher-Snedecor diffusion

The Fisher-Snedecor diffusion satisfies the SDE

$$dX_t = -\theta\left(X_t - \frac{\beta}{\beta-2}\right)dt + \sqrt{\frac{4\theta}{\gamma(\beta-2)}X_t(\gamma X_t+\beta)}\,dW_t, \quad t \geq 0$$

with $\theta > 0$ and has invariant density

$$\mathfrak{fs}(x) = \frac{\beta^{\frac{\beta}{2}}}{B\left(\frac{\gamma}{2},\frac{\beta}{2}\right)}\frac{(\gamma x)^{\frac{\gamma}{2}-1}}{(\gamma x+\beta)^{\frac{\gamma}{2}+\frac{\beta}{2}}}\,\gamma\,\mathrm{I}_{\langle 0,\infty\rangle}(x) \tag{9}$$

with parameters $\gamma > 0$ and $\beta > 2$, where the latter requirement ensures the existence of the stationary mean $\beta/(\beta-2)$.

Theorem 2. *The transition density of fractional FS diffusion is given by*

$$\begin{aligned} p_\alpha(x,t;x_0) &= \sum_{n=0}^{\lfloor\frac{\beta}{4}\rfloor}\mathfrak{fs}(x)\,F_n(x_0)\,F_n(x)\,\mathcal{E}_\alpha(-\lambda_n t^\alpha) \\ &+ \frac{\mathfrak{fs}(x)}{\pi}\int_{\frac{\theta\beta^2}{8(\beta-2)}}^{\infty}\mathcal{E}_\alpha(-\lambda t^\alpha)\,a(\lambda)\,f_1(x_0,-\lambda)\,f_1(x,-\lambda)\,d\lambda, \end{aligned} \tag{10}$$

where F_n are normalized Fisher-Snedecor polynomials, λ_n are their eigenvalues, $a(\lambda)$ is a constant depending on λ and f_1 is the solution of the corresponding Sturm-Liouville equation.

For proof and details see [4].

5 Stationary distributions of the fractional reciprocal gamma and Fisher-Snedecor diffusions

By $p_\alpha(x,t)$ denote the density of $X_\alpha(t)$, by $p(x,t)$ the density of $X(t)$ and let f be the density of initial state $X_\alpha(0)$. Now, by the definition of transition density it follows

$$p_\alpha(x,t) = \int_0^\infty p_\alpha(x,t;y) f(y) dy.$$

If we assume that the initial distribution is concentrated in one point, i.e. if $f(y) = \delta(x_0)$ we obtain

$$p_\alpha(x,t) = p_\alpha(x,t;x_0)$$

and since for fractional FS and RG diffusion, transition densities $p_\alpha(x,t;x_0)$ are given via (8) and (10), one can show that

$$p_\alpha(x,t) \to m(x) \ as \ t \to \infty, \tag{11}$$

where m is FS stationary distribution in fractional FS diffusion case, and RG stationary distribution in fractional RG case.
In fact, even without the assumption on the concentrated initial state, one can prove the statement, for details we refer to [4].
Also, obsverve that

$$p_\alpha(x,t) \to p(x,t) \ as \ \alpha \to 1. \tag{12}$$

6 Correlation structure of fractional Pearson diffusions

Stationary Pearson diffusion $X(t)$ such that the stationary distribution has finite second moment has the correlation function given by

$$\mathrm{Corr}\,[X(t),\, X(s)] = \exp(-\theta|t-s|), \tag{13}$$

where θ is the autocorrelation parameter. Since the autocorrelation function (13) falls off exponentially, Pearson diffusions exhibit short-range dependence.

We say that fractional Pearson diffusion $X_\alpha(t)$ defined by (4) is in the steady state if it starts from its invariant distribution with the density m. Then the autocorrelation function of $X_\alpha(t) = X(E_t)$ is given by

$$\mathrm{Corr}\,[X_\alpha(t),\, X_\alpha(s)] = \mathcal{E}_\alpha(-\theta t^\alpha) + \frac{\theta\alpha t^\alpha}{\Gamma(1+\alpha)} \int\limits_0^{s/t} \frac{\mathcal{E}_\alpha(-\theta t^\alpha (1-z)^\alpha)}{z^{1-\alpha}} dz \tag{14}$$

for $t \geq s > 0$. The tehnique to prove this fact can be found in [2].
Observe that (14) implies the long-range dependence of the fractional diffusion $X_\alpha(t)$, since the autocorrelation function (14) falls off like power law with exponent $\alpha \in (0,1)$.

7 Simulation results

Simulation results are based on the algorithm introduced in [6]. Basically, idea is to seperately simulate trajectory of the inverse of the stable subordinator and trajectory of the non-fractional diffusion. Afterwards, by linear interpolation one gets trajectory of the fractional diffusion. This algorithm perfectly fits our setting, since we define fractional Pearson diffusion as a composition of the non-fractional Pearson diffusion and the inverse of the stable subordinator (which are assume to be independent).
Trajectories of such simulated fractional RG and FS diffusion are given in Figure 1, where the difference between non-fractional and fractional diffusions can be clearly seen. Unlike non-fractional diffusions, fractional diffusions have long resting periods of time due to change of time via inverse of the stable subordinator E_t.

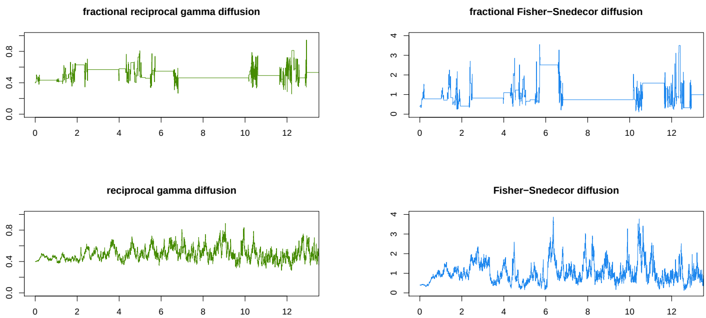

Figure 1: Sample paths of the fractional/non-fractional RG and FS diffusions with parameters $\gamma = 10$, $\beta = 20$, $\theta = 0.01$ and $\alpha = 0.7$, based on 10000 points with inital state $X_0 = 0.4$.

Next, we illustrate that density of fractional diffusion approach the stationary density as explained in Section 5. We simulated 1000 trajectories of the fractional RG diffusion and estimated densities at times $t = 0.02$, $t = 0.2$ and $t = 2$, see figure 2. Comparing densities $p_\alpha(x,t)$ and $p(x,t)$, we clearly observe slower approaching to the stationary density in fractional case. Autocorrelation function (14) of the fractional diffusion, in comparison with the autocorrelation function (13) of the non-fractional diffusion which decays exponentially fast, decays much slower, i.e. in polynomial rate. This is illustrated in Figure 3.

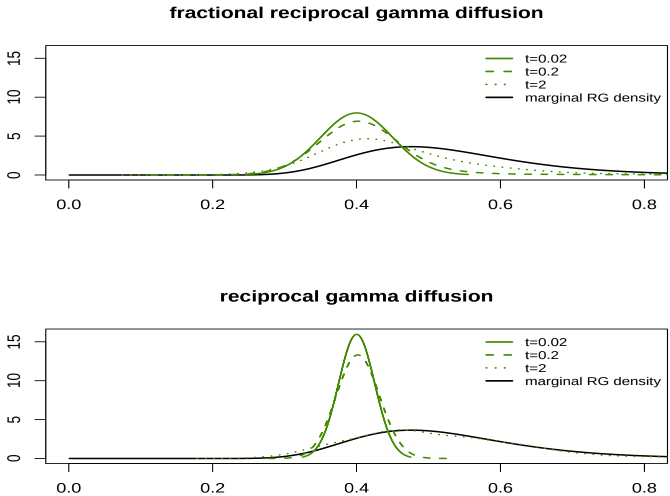

Figure 2: Estimated densities $p_\alpha(x,t)$ and $p(x,t)$ for reciprocal gamma diffusion with parameters $\gamma = 10$, $\beta = 20$, $\theta = 0.01$ and $\alpha = 0.7$, based on 1000 trajectories with inital state $X_0 = 0.4$.

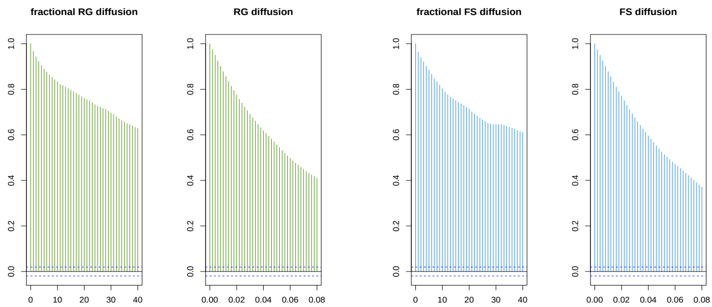

Figure 3: Estimated autocorrelation function of fractional/non fractional RG and FS diffusions with parameters $\gamma = 10$, $\beta = 20$, $\theta = 0.01$ and $\alpha = 0.7$, based on 10000 points with inital state $X_0 = 0.4$.

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