Nonparametric estimation of gradual change points in the jump behaviour of an Itō semimartingale

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In applications the properties of a stochastic feature often change gradually rather than abruptly, that is: after a constant phase for some time they slowly start to vary. In this paper we discuss the localisation of a gradual change point in the jump characteristic of a discretely observed Itō semimartingale. We propose a new measure of time variation for the jump behaviour of the process. Based on weak convergence of a suitable stochastic process we derive an estimator for the first point in time where the jump characteristic changes.

Keywords: Lévy measure, jump compensator, empirical processes, weak convergence, gradual changes

1 Introduction

Stochastic processes in continuous time are widely used in science nowadays, as they allow for a flexible modeling of the evolution of various real-life phenomena over time. Speaking of mathematical finance, of particular interest is the family of semimartingales, which is theoretically appealing as it satisfies a certain condition on the absence of arbitrage in financial markets and yet is rich enough to reproduce stylized facts from empirical finance such as volatility clustering, leverage effects or jumps. For this reason, the development of statistical tools modeled by discretely observed Itō semimartingales has been a major topic over the last years, both regarding the estimation of crucial quantities used for model calibration purposes and with a view on tests to check whether a certain model fits the data well. For a detailed overview of the state of the art we refer to the recent monographs by [4] and [1].

In the following, we are interested in the evolution of the jump behaviour over time in a completely non-parametric setting where we assume only structural

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conditions on the characteristic triplet of the underlying Itō semimartingale. To be precise, let $X = (X_t)_{t \geq 0}$ be an Itō semimartingale with a decomposition

$$X_t = X_0 + \int_0^t b_s \, ds + \int_0^t \sigma_s \, dW_s + \int_0^t \int_{\mathbb{R}} z 1_{\{|z| \leq 1\}} (\mu - \bar{\mu})(ds, dz) + \int_0^t \int_{\mathbb{R}} z 1_{\{|z| > 1\}} \mu(ds, dz), \quad (1.1)$$

where W is a standard Brownian motion, μ is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$, and the predictable compensator $\bar{\mu}$ satisfies $\bar{\mu}(ds, dz) = ds \, \nu_s(dz)$. The main quantity of interest is the kernel ν_s which controls the number and the size of the jumps around time s. In [2] the authors are interested in the detection of abrupt changes in the jump measure of X. Based on high-frequency observations $X_{i\Delta_n}$, $i = 0, \ldots, n$, with $\Delta_n \to 0$ they construct a test for a constant ν against the alternative

$$\nu_s^{(n)} = 1_{\{s < \lfloor n\theta_0 \rfloor \Delta_n\}} \nu_1 + 1_{\{s \geq \lfloor n\theta_0 \rfloor \Delta_n\}} \nu_2.$$

In the sequel, we will deal with gradual (smooth, continuous) changes of ν_s which basically means that ν_s is a non-constant function in $s \in \mathbb{R}_+$. We discuss how and how well the first point in time where the jump behaviour changes (gradually) can be estimated. To this end, we introduce the formal setup in Section 2 where we also define a measure of time variation which is used to detect changes in the jump characteristic. Section 3 is concerned with weak convergence of a standardized version of an estimator for this measure. In Section 4 we use this result to derive an estimator of the first change point for the jump behaviour. The proofs of the results presented in this paper can be found in [3].

2 The basic assumptions and a measure of gradual changes

In the sequel let $X^{(n)} = (X_t^{(n)})_{t \geq 0}$ be an Itō semimartingale of the form (1.1) with characteristic triplet $(b_s^{(n)}, \sigma_s^{(n)}, \nu_s^{(n)})$ for each $n \in \mathbb{N}$. We are interested in investigating gradual changes in the evolution of the jump behaviour and we assume throughout this paper that there is a driving law behind this evolution which is common for all $n \in \mathbb{N}$. Formally, we introduce a transition kernel $g(y, dz)$ from $([0,1], \mathbb{B}([0,1]))$ into $(\mathbb{R}, \mathbb{B})$ such that

$$\nu_s^{(n)}(dz) = g\Big(\frac{s}{n\Delta_n}, dz\Big)$$

for $s \in [0, n\Delta_n]$. This transition kernel shall be an element of the set $\mathcal{G}$ to be defined below. Throughout the paper $\mathbb{B}(A)$ denotes the trace σ-algebra on a set $A \subset \mathbb{R}$ with respect to the Borel σ-algebra $\mathbb{B}$ of $\mathbb{R}$.

Assumption 1. *Let $\mathcal{G}$ denote the set of all transition kernels $g(\cdot, dz)$ from $([0,1], \mathbb{B}([0,1]))$ into $(\mathbb{R}, \mathbb{B})$ such that*

(1) For each $y \in [0,1]$ the measure $g(y, dz)$ does not charge $\{0\}$, i.e. $g(y, \{0\}) = 0$.

(2) The function $y \mapsto \int (1 \wedge z^2) g(y, dz)$ is bounded on the interval $[0,1]$.

(3) If

$$\mathcal{I}(z) := \begin{cases} [z, \infty), & \text{for } z > 0 \\ (-\infty, z], & \text{for } z < 0 \end{cases}$$

denotes one-sided intervals and

$$g(y,z) := g(y, \mathcal{I}(z)) = \int_{\mathcal{I}(z)} g(y, dx); \quad (y,z) \in [0,1] \times \mathbb{R} \setminus \{0\},$$

then for every $z \in \mathbb{R} \setminus \{0\}$ there exists a finite set $M^{(z)} = \{t_1^{(z)}, \ldots, t_{n_z}^{(z)} \mid n_z \in \mathbb{N}\} \subset [0,1]$, such that the function $y \mapsto g(y,z)$ is continuous on $[0,1] \setminus M^{(z)}$.

(4) For each $y \in [0,1]$ the measure $g(y, dz)$ is absolutely continuous with respect to the Lebesgue measure with density $z \mapsto h(y,z)$, where the measurable function $h \colon ([0,1] \times \mathbb{R}, \mathbb{B}([0,1]) \otimes \mathbb{B}) \to (\mathbb{R}, \mathbb{B})$ is continuously differentiable with respect to $z \in \mathbb{R} \setminus \{0\}$ for fixed $y \in [0,1]$. The function $h(y,z)$ and its derivative will be denoted by $h_y(z)$ and $h'_y(z)$, respectively. Furthermore, we assume for each $\varepsilon > 0$ that

$$\sup_{y \in [0,1]} \sup_{z \in M_\varepsilon} \Big(h_y(z) + |h'_y(z)| \Big) < \infty,$$

where $M_\varepsilon = (-\infty, -\varepsilon] \cup [\varepsilon, \infty)$.

In order to investigate gradual changes in the jump behaviour of the underlying process we follow [5] and consider a measure of time variation for the jump behaviour which is defined by

$$D(\zeta, \theta, z) := \int_0^\zeta g(y,z) dy - \frac{\zeta}{\theta} \int_0^\theta g(y,z) dy, \tag{2.1}$$

where $(\zeta, \theta, z) \in C \times \mathbb{R} \setminus \{0\}$ and

$$C := \{(\zeta, \theta) \in [0,1]^2 \mid \zeta \le \theta\}. \tag{2.2}$$

Here and throughout this paper we use the convention $\frac{0}{0} := 1$. The time varying measure defined in (2.1) is indeed suitable for the detection of gradual changes in the jump characteristic of the underlying process, because one can show that the jump behaviour corresponding to the first $\lfloor n\theta \rfloor$ observations is identical for some $\theta \in [0,1]$ if and only if $D(\zeta, \theta, z) \equiv 0$ for all $0 \le \zeta \le \theta$ and $z \in \mathbb{R} \setminus \{0\}$ (see [3]).

We conclude this section with the main assumption for the characteristics of an Itō semimartingale which will be used throughout this paper.

Assumption 2. *For each $n \in \mathbb{N}$ let $X^{(n)}$ denote an Itō semimartingale of the form* (1.1) *with characteristics $(b_s^{(n)}, \sigma_s^{(n)}, \nu_s^{(n)})$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that satisfies*

(a) There exists a $g \in \mathcal{G}$ such that

$$\nu_s^{(n)}(dz) = g\Big(\frac{s}{n\Delta_n}, dz\Big)$$

holds for all $s \in [0, n\Delta_n]$ and all $n \in \mathbb{N}$.

(b) The drift $b_s^{(n)}$ and the volatility $\sigma_s^{(n)}$ are predictable processes and satisfy

$$\sup_{n\in\mathbb{N}} \sup_{s\in\mathbb{R}_+} \Big(\mathbb{E}|b_s^{(n)}|^\alpha \vee \mathbb{E}|\sigma_s^{(n)}|^p\Big) < \infty,$$

for some $p > 2$, with $\alpha = 3p/(p+4)$.

(c) The observation scheme $\{X_{i\Delta_n}^{(n)} \mid i = 0, \ldots, n\}$ satisfies

$$\Delta_n \to 0, \quad n\Delta_n \to \infty, \quad \text{and} \quad n\Delta_n^{1+\tau} \to 0,$$

for $\tau = (p-2)/(p+1) \in (0,1)$.

3 An estimator for the measure of time variation and weak convergence

In order to estimate the measure of time variation introduced in (2.1) we use the sequential empirical tail integral process defined by

$$U_n(\theta, z) = \frac{1}{n\Delta_n} \sum_{j=1}^{\lfloor n\theta \rfloor} 1_{\{\Delta_j^n X^{(n)} \in \mathcal{I}(z)\}},$$

where $\Delta_j^n X^{(n)} = X_{j\Delta_n}^{(n)} - X_{(j-1)\Delta_n}^{(n)}$, $\theta \in [0,1]$ and $z \in \mathbb{R} \setminus \{0\}$. An estimate for the measure of time variation defined in (2.1) is then given by

$$\mathbb{D}_n(\zeta,\theta,z) := U_n(\zeta,z) - \frac{\zeta}{\theta} U_n(\theta,z), \quad (\zeta,\theta,z) \in C \times \mathbb{R} \setminus \{0\}, \tag{3.1}$$

where the set C is defined in (2.2). The following theorem establishes consistency of $\mathbb{D}_n$ as it shows weak convergence of the process

$$\mathbb{H}_n(\zeta,\theta,z) := \sqrt{n\Delta_n}(\mathbb{D}_n(\zeta,\theta,z) - D(\zeta,\theta,z)). \tag{3.2}$$

with values in $\ell^\infty(B_\varepsilon)$, where $B_\varepsilon = C \times M_\varepsilon$.

Theorem 1. *If Assumption 2 is satisfied, then the process $\mathbb{H}_n$ defined in* (3.2) *satisfies $\mathbb{H}_n \rightsquigarrow \mathbb{H}$ in $\ell^\infty(B_\varepsilon)$ for any $\varepsilon > 0$, where $\mathbb{H}$ is a tight mean zero Gaussian process with covariance function*

$$\begin{aligned}
&\mathrm{Cov}(\mathbb{H}(\zeta_1,\theta_1,z_1),\mathbb{H}(\zeta_2,\theta_2,z_2)) = \\
&= \int_0^{\zeta_1\wedge\zeta_2} g(y,\mathcal{I}(z_1)\cap\mathcal{I}(z_2))dy - \frac{\zeta_1}{\theta_1}\int_0^{\zeta_2\wedge\theta_1} g(y,\mathcal{I}(z_1)\cap\mathcal{I}(z_2))dy \\
&\quad - \frac{\zeta_2}{\theta_2}\int_0^{\zeta_1\wedge\theta_2} g(y,\mathcal{I}(z_1)\cap\mathcal{I}(z_2))dy + \frac{\zeta_1\zeta_2}{\theta_1\theta_2}\int_0^{\theta_1\wedge\theta_2} g(y,\mathcal{I}(z_1)\cap\mathcal{I}(z_2))dy.
\end{aligned}$$

4 A consistent estimator for the gradual change point

If one defines

$$\mathcal{D}^{(\varepsilon)}(\theta) := \sup_{|z|\geq\varepsilon} \sup_{0\leq\zeta\leq\theta'\leq\theta} |D(\zeta,\theta',z)|,$$

for some pre-specified constant $\varepsilon > 0$, one can characterize the existence of a change point as follows: There exists a gradual change in the behaviour of the jumps larger than ε of the process (1.1) if and only if $\mathcal{D}^{(\varepsilon)}(1) > 0$. Our aim is to construct an estimator for the first point where the jump behaviour changes (gradually). For this purpose we define

$$\theta_0^{(\varepsilon)} := \inf\left\{\theta \in [0,1] \mid \mathcal{D}^{(\varepsilon)}(\theta) > 0\right\},$$

where we set $\inf \emptyset := 1$. We call $\theta_0^{(\varepsilon)}$ the change point of the jumps larger than ε of the underlying process (1.1). Intuitively, the estimation of $\theta_0^{(\varepsilon)}$ becomes more difficult the flatter the curve $\theta \mapsto \mathcal{D}^{(\varepsilon)}(\theta)$ is at $\theta_0^{(\varepsilon)}$. Therefore, we describe the curvature of $\theta \mapsto \mathcal{D}^{(\varepsilon)}(\theta)$ by a local polynomial behaviour of the function $\mathcal{D}^{(\varepsilon)}(\theta)$ for values $\theta > \theta_0^{(\varepsilon)}$. More precisely, we assume throughout this section that $\theta_0^{(\varepsilon)} < 1$ and that there exist constants $\lambda, \eta, \varpi, c^{(\varepsilon)} > 0$ such that $\mathcal{D}^{(\varepsilon)}$ admits an expansion of the form

$$\mathcal{D}^{(\varepsilon)}(\theta) = c^{(\varepsilon)}\big(\theta - \theta_0^{(\varepsilon)}\big)^{\varpi} + \aleph(\theta) \tag{4.1}$$

for all $\theta \in [\theta_0^{(\varepsilon)}, \theta_0^{(\varepsilon)} + \lambda]$, where the remainder term satisfies $|\aleph(\theta)| \leq K\big(\theta - \theta_0^{(\varepsilon)}\big)^{\varpi+\eta}$ for some $K > 0$. By Theorem 1 the process $\mathbb{D}_n(\zeta, \theta, z)$ from (3.1) is a consistent estimator of $D(\zeta, \theta, z)$. Therefore we set

$$\mathbb{D}_n^{(\varepsilon)}(\theta) := \sup_{|z|\geq\varepsilon} \sup_{0\leq\zeta\leq\theta'\leq\theta} |\mathbb{D}_n(\zeta, \theta', z)|.$$

The construction of an estimator for $\theta_0^{(\varepsilon)}$ utilizes the fact that $(n\Delta_n)^{1/2}\mathbb{D}_n^{(\varepsilon)}(\theta) \to \infty$ in probability for any $\theta \in (\theta_0^{(\varepsilon)}, 1]$. Moreover, for $\theta \in [0, \theta_0^{(\varepsilon)}]$ we have $(n\Delta_n)^{1/2}\mathbb{D}_n^{(\varepsilon)}(\theta) = O_{\mathbb{P}}(1)$ since this quantity converges weakly. Therefore, we consider the statistic

$$r_n^{(\varepsilon)}(\theta) := \mathbb{1}_{\{(n\Delta_n)^{1/2}\mathbb{D}_n^{(\varepsilon)}(\theta)\leq\varkappa_n\}},$$

for a deterministic sequence $\varkappa_n \to \infty$. From the previous discussion we expect

$$r_n^{(\varepsilon)}(\theta) \to \begin{cases} 1, & \text{if } \theta \leq \theta_0^{(\varepsilon)} \\ 0, & \text{if } \theta > \theta_0^{(\varepsilon)} \end{cases}$$

in probability if the threshold level $\varkappa_n$ is chosen appropriately. Consequently, we define the estimator for the change point by

$$\hat{\theta}_n^{(\varepsilon)} = \hat{\theta}_n^{(\varepsilon)}(\varkappa_n) := \int_0^1 r_n^{(\varepsilon)}(\theta) d\theta.$$

The following result establishes consistency of the estimator $\hat{\theta}_n^{(\varepsilon)}$ under rather mild assumptions on the sequence $(\varkappa_n)_{n\in\mathbb{N}}$.

Theorem 2. *If Assumption 2 is satisfied, $\theta_0^{(\varepsilon)} < 1$, and (4.1) holds for some $\varpi > 0$, then*

$$\hat{\theta}_n^{(\varepsilon)} - \theta_0^{(\varepsilon)} = O_{\mathbb{P}}\Big(\big(\frac{\varkappa_n}{\sqrt{n\Delta_n}}\big)^{1/\varpi}\Big),$$

for any sequence $\varkappa_n \to \infty$ with $\varkappa_n/\sqrt{n\Delta_n} \to 0$.

Theorem 2 makes the heuristic argument above more precise. A lower degree of smoothness in $\theta_0^{(\varepsilon)}$ yields a better rate of convergence of the estimator. Moreover, the slower the threshold level $\varkappa_n$ converges to infinity the better the rate of convergence. In [3] the authors discuss a data-driven choice of the threshold $\varkappa_n$ for which the probability for over- and underestimation of $\theta_0^{(\varepsilon)}$ can be controlled.

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