Measurement of phase space density evolution in MICE Step IV

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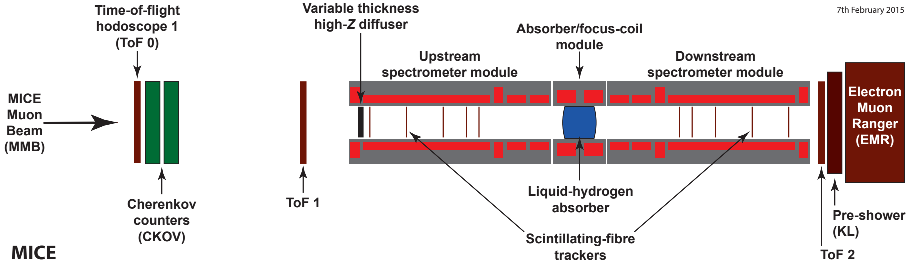
Experimental apparatus at present (Step IV)

All the detectors are installed and **working**

- Three time-of-flight (TOF) detector stations
- Two Cherenkov counters and a downstream calorimetry module
- Two scintillating-fibre trackers

Part of the cooling channel (no RF yet)

- Two Spectrometer Solenoids (SS): 5 coils each
  - The two downstream match coils are currently **not turned on**
- An Absorber Focus Coil (AFC): 2 coils + absorber

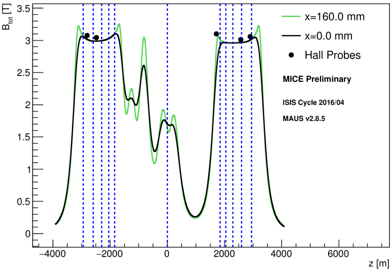


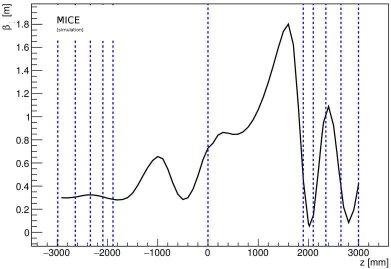
Beam cooling setting optimization

Suitable optics have been found using two approaches to conjointly optimize transmission and cooling performance:

- Linear optics, scan in the parameter space of magnet currents;
- Genetic algorithm, best sets of optics bear the next generation, penalize transmission loss and encourage emittance reduction.

The **bottom lattice** setting is expected to have one of the best cooling performance – transmission trade-off and is **presented in the following**



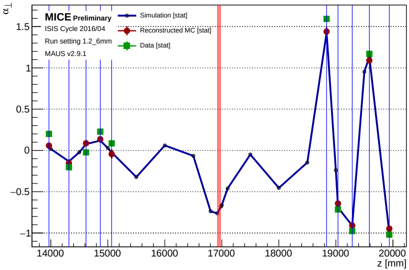


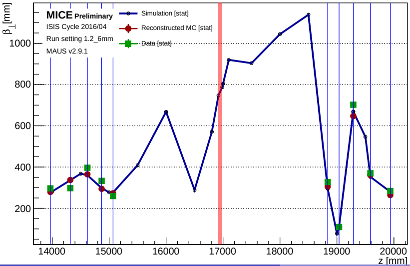
Reproduction of optical functions in the simulation

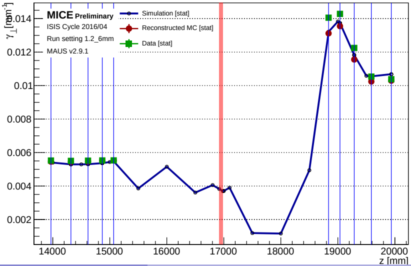
Reliable reproduction of the data in the simulation for

- ∼6 mm input beam
- 140 MeV/c central momentum
- -0.68 central $\alpha_\perp$
- 787 mm central $\beta_\perp$

→ Excellent tool to systematically test novel density methods





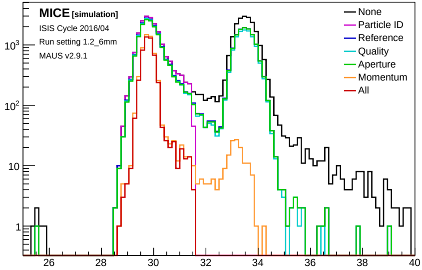


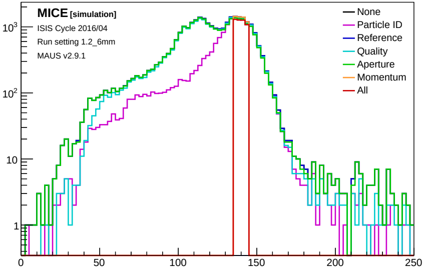
Particle selection

Series of cuts applied to both data and simulation:

- Muon tagging using TOF01 (Particle ID)
- Upstream reference plane hit (Reference)
- Good track reconstruction quality (Quality)
- Track within the tracker fiducial (Aperture)
- Longit. momentum $\in [135, 145]$ MeV/c (Momentum)

} **All**





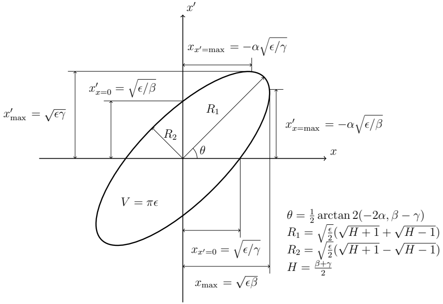
Transverse normalised RMS emittance

4D normalised RMS emittance:

$$\boxed{\epsilon_n = \frac{1}{m}|\Sigma|^{\frac{1}{4}}}, \qquad (1)$$

with $|\Sigma|$ the determinant of the co-variance matrix defined as

$$\Sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xp_x} & \sigma_{xy} & \sigma_{xp_y} \\ \sigma_{p_x x} & \sigma_{p_x p_x} & \sigma_{p_x y} & \sigma_{p_x p_y} \\ \sigma_{yx} & \sigma_{yp_x} & \sigma_{yy} & \sigma_{yp_y} \\ \sigma_{p_y x} & \sigma_{p_y p_x} & \sigma_{p_y y} & \sigma_{p_y p_y} \end{pmatrix}. \qquad (2)$$

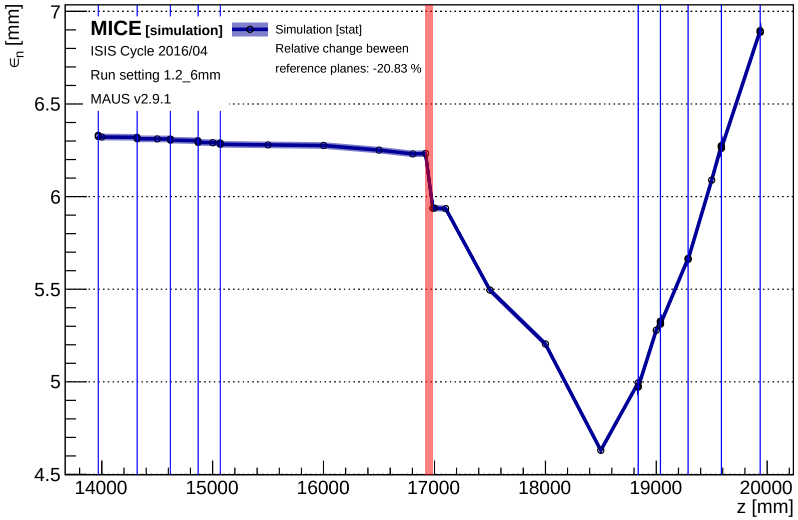


The RMS emittance is directly related to the **volume of the RMS ellipsoid** through $\epsilon_n = \sqrt{2V_{\text{RMS}}}/(m\pi)$ and as such is the most common probe of average phase space density:

$$\rho_{\text{RMS}} = \frac{N}{V_{\text{RMS}}} = \frac{N}{\frac{1}{2}m^2\pi^2\epsilon_n^2} = \frac{N}{\frac{1}{2}\pi^2|\Sigma|^{\frac{1}{2}}} \quad [\text{mm}^{-2}(\text{MeV}/c)^{-2}]. \qquad (3)$$

$\rightarrow$ It follows from Liouville's theorem that the phase space volume should be conserved

Emittance evolution



Power of density estimation

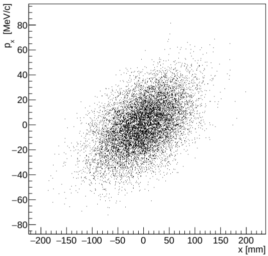
The emittance plot exhibits two obvious **challenges**:

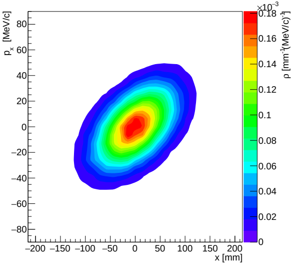
- transmission losses yield apparent emittance reduction;
- filamentation in SSD yield apparent emittance growth.

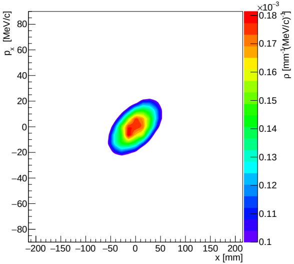
The key to solving both problems lies in **density estimation**:

- Estimate density in the transverse 4D phase space (*center figure*);
- Select an **identical fraction** of the beam upstream and downstream from within the **densest area of the space** (*right figure*);
- Define cooling figure of merits on these subsamples;

→ The core, unlike the tails, is **transmitted** and **linear**.







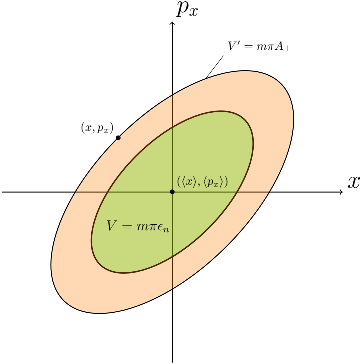
Transverse single-particle amplitude

Single particle amplitude is defined as

$$A_\perp = \epsilon_n \mathbf{u}^T \Sigma^{-1} \mathbf{u} \qquad (4)$$

with $\mathbf{u} = \mathbf{v} - \boldsymbol{\mu}$, the centered phase space vector, $\mathbf{v} = (x, p_x, y, p_y)$, of the particle.

- It is related to the **volume** of an ellipse, which is similar to the RMS ellipse, going through $\mathbf{v}$.
- Amplitude follows a $\boldsymbol{\chi^2}$ **distribution** with d degrees of freedom



Particle amplitude provides a density estimate in every input point

$$\rho(\boldsymbol{v}_i) = \frac{1}{(2\pi)^2|\Sigma|^{\frac{1}{2}}} \exp\left[-\mathbf{u}^T\Sigma^{-1}\mathbf{u}/2\right] = \boxed{\frac{1}{4\pi^2 m^2 \epsilon_n^2} \exp\left[-\frac{A_\perp}{2\epsilon_n}\right]}. \qquad (5)$$

→ Allows for the selection of a **high density core**

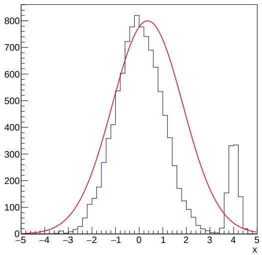
Amplitude reconstruction

In the case of non-linear beams, special care must be taken in the reconstruction of amplitude as tails significantly bias the covariance matrix

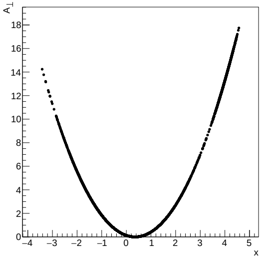
Optimal procedure for amplitude reconstruction:

- Compute Σ and μ for the whole sample;

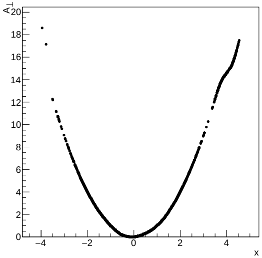
1. Calculate all the particle amplitudes $A_\perp^i$;
2. Register the highest amplitude in the distribution;
3. Update Σ and μ by removing the highest amplitude point;
4. Iterate from 1.



Test Gaussian + outliers

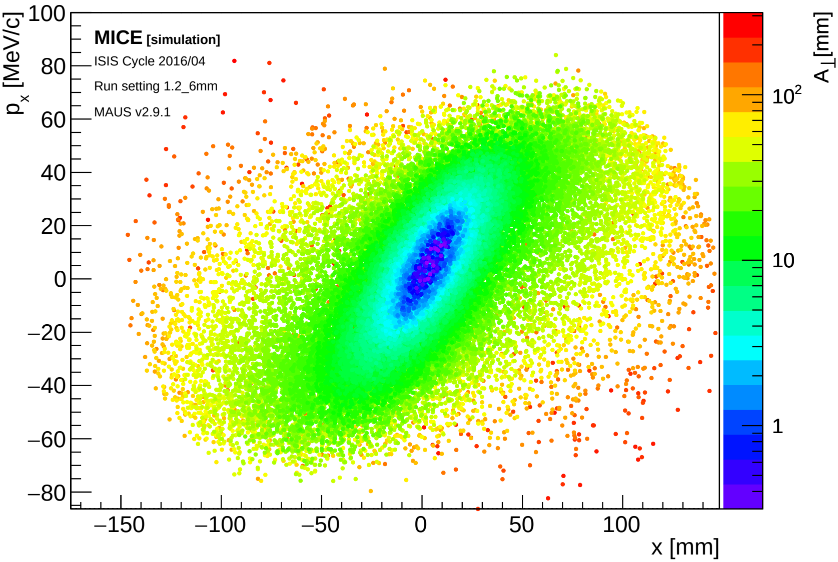


Regular amplitudes (biased)

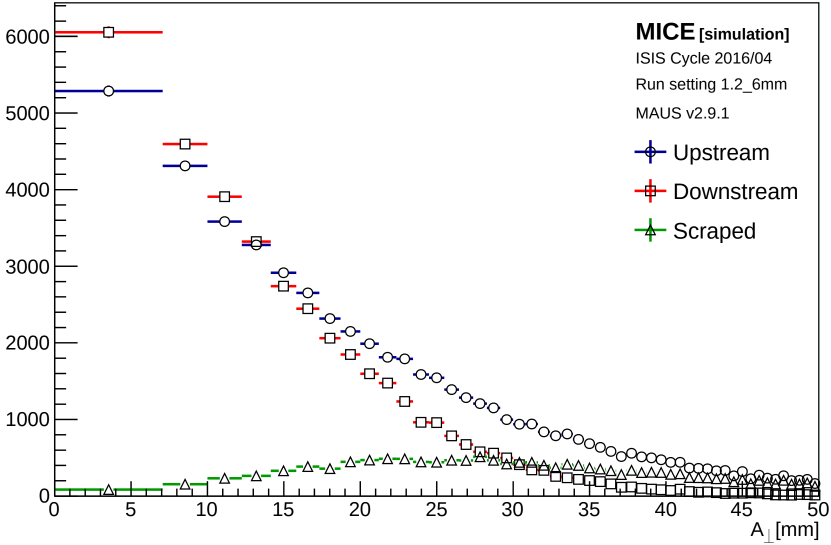


Corrected amplitudes

Amplitudes at TKD station 5



Amplitude distribution evolution



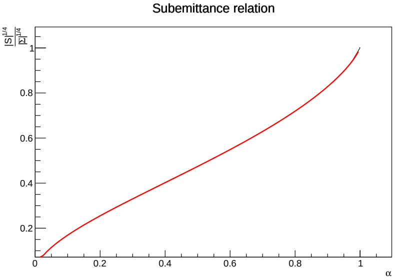
Subemittance definition and properties

The $\boldsymbol{\alpha}$**-subemittance**, $\boldsymbol{e_\alpha}$, is defined as the emittance of the core fraction α of the parent beam. For a truncated 4D Gaussian beam of covariance matrix S, it satisfies

$$\frac{e_\alpha}{\epsilon_n} = \frac{|S|^{\frac{1}{4}}}{|\Sigma|^{\frac{1}{4}}} = \frac{1}{2\boldsymbol{\alpha}}\gamma\left(3, \frac{\boldsymbol{R}^2}{2}\right), \qquad (6)$$

$$R^2 = Q_{\chi^2_4}(\boldsymbol{\alpha}).$$

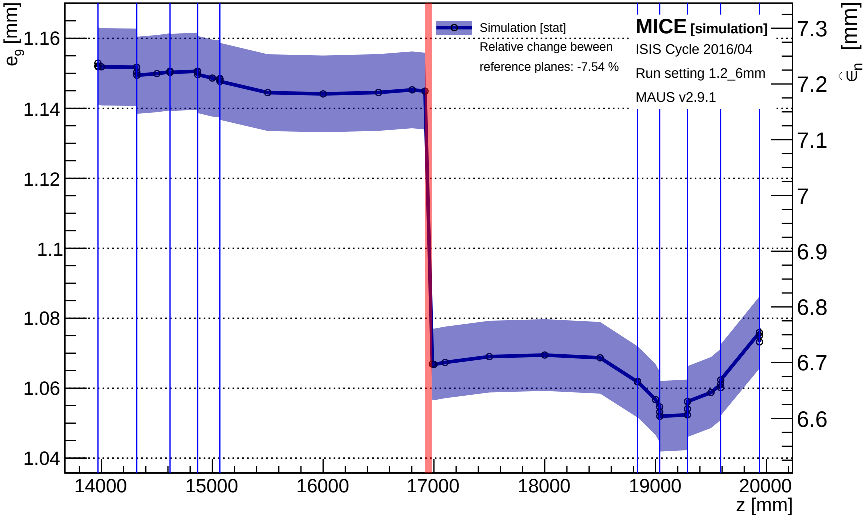
$$\rightarrow \boxed{\frac{e_\alpha^{\text{out}} - e_\alpha^{\text{in}}}{e_\alpha^{\text{in}}} = \frac{\epsilon_n^{\text{out}} - \epsilon_n^{\text{in}}}{\epsilon_n^{\text{in}}}}. \qquad (7)$$



The statistical uncertainty carried by this measurement is identical to that of the emittance, scaled by the fraction α as

$$\frac{\sigma_{e_\alpha}}{e_\alpha} = \frac{1}{\sqrt{\alpha}}\frac{\sigma_{\epsilon_n}}{\epsilon_n} = \sqrt{\frac{2}{\alpha N d}}. \qquad (8)$$

Subemittance evolution

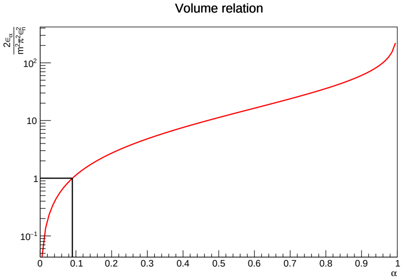


Fractional emittance definition and properties

The **α-fractional emittance**, $\boldsymbol{\epsilon_\alpha}$, is defined as the **phase space volume** occupied by the core fraction α of the parent beam. For a truncated 4D Gaussian beam of covariance matrix S, it satisfies

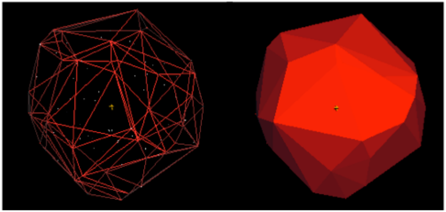
$$\epsilon_\alpha = \frac{1}{2}m^2\pi^2\boldsymbol{\epsilon_n^2}R^4 = V_{\text{RMS}}\boldsymbol{R}^4, \qquad (9)$$
$$R^2 = Q_{\chi^2_4}(\boldsymbol{\alpha}).$$

$$\rightarrow \boxed{\frac{\epsilon_\alpha^{\text{out}} - \epsilon_\alpha^{\text{in}}}{\epsilon_\alpha^{\text{in}}} \simeq \mathbf{2}\frac{\epsilon_n^{\text{out}} - \epsilon_n^{\text{in}}}{\epsilon_n^{\text{in}}}} \qquad (10)$$

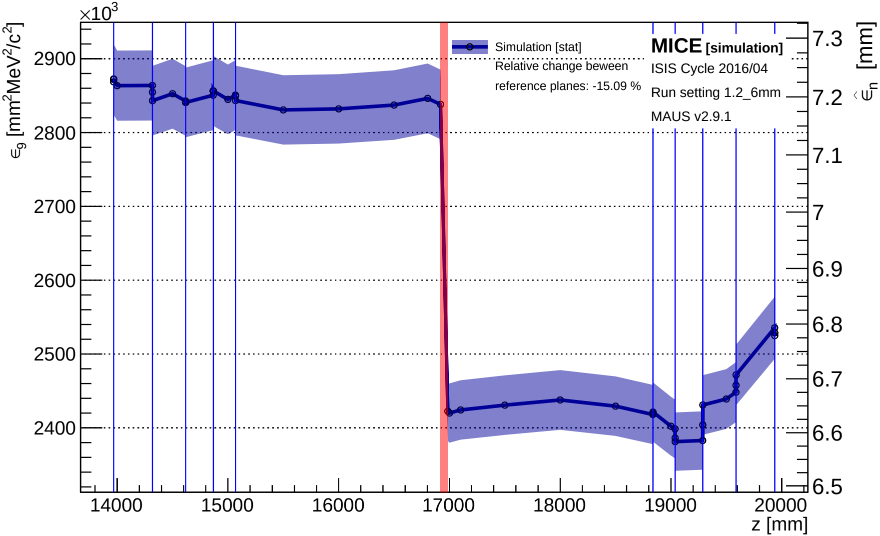


In 4D, a fraction α of **9%** yields the volume of the **RMS ellipsoid**, V_{RMS}

The **convex hull** is a prime candidate for volume reconstruction. It computes the smallest volume that contains the core αN points.



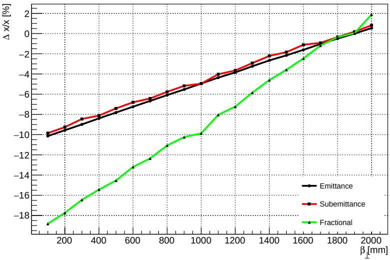
Phase space volume evolution

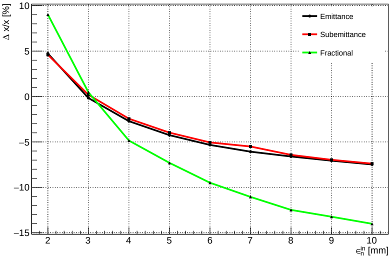


Toy analysis of fractional quantities

A toy analysis (Gaussian input beam, toy absorber) shows:

- The **same relative change** is seen in the RMS emittance and all of the fractional quantities, for any fraction
- The change in fractional quantities exhibit the **same relation** with $\beta_\perp$ and the input emittance, ϵ_i
- The fractional quantities are **more robust** against losses and non-linearities as the tails do not influence their measurement





→ Plots produced for a core 9 % selection, i.e. size of the RMS ellipse

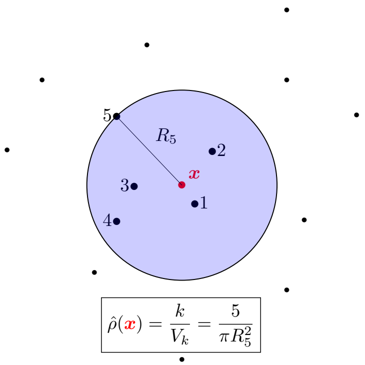
Non-parametric density estimation: kNN

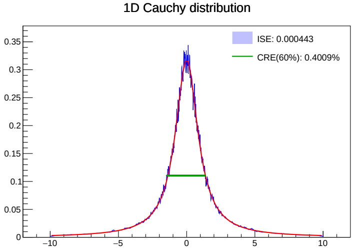
For a given point $\boldsymbol{x}$, find the $\boldsymbol{k}$ **closest points** in the input cloud. Find the distance R_k to the k^{th} point and compute the 4D local density estimate as

$$\rho(\boldsymbol{x}) = \frac{k}{\mathcal{V}_k} = \frac{2k}{\pi^2 R_k^4}, \qquad (11)$$

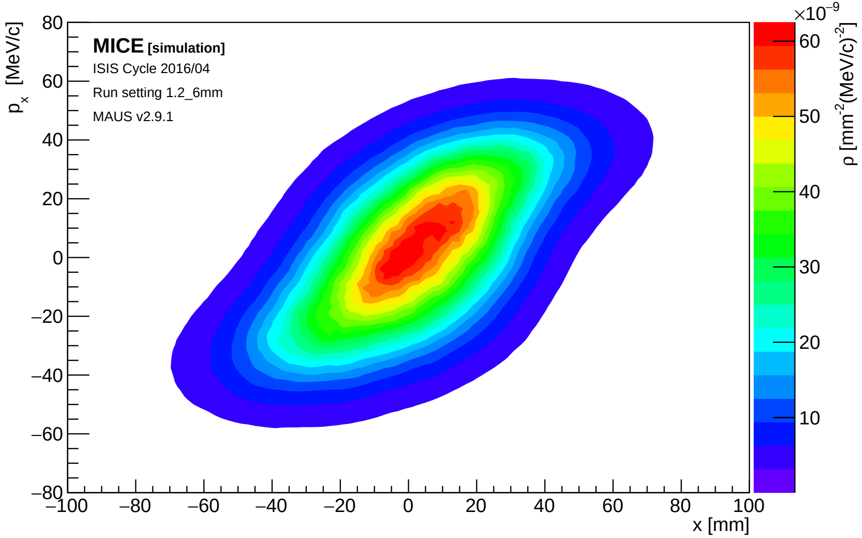
with $\mathcal{V}_k$ the volume of the 4-ball centred in $\boldsymbol{x}$ of radius R_k.

- Rule of thumb choice of $\boldsymbol{k = \sqrt{N}}$ yields quasi-optimal results for a broad array of distributions.
- Right plot shows great agreement between Cauchy distribution (red) and estimation (blue).

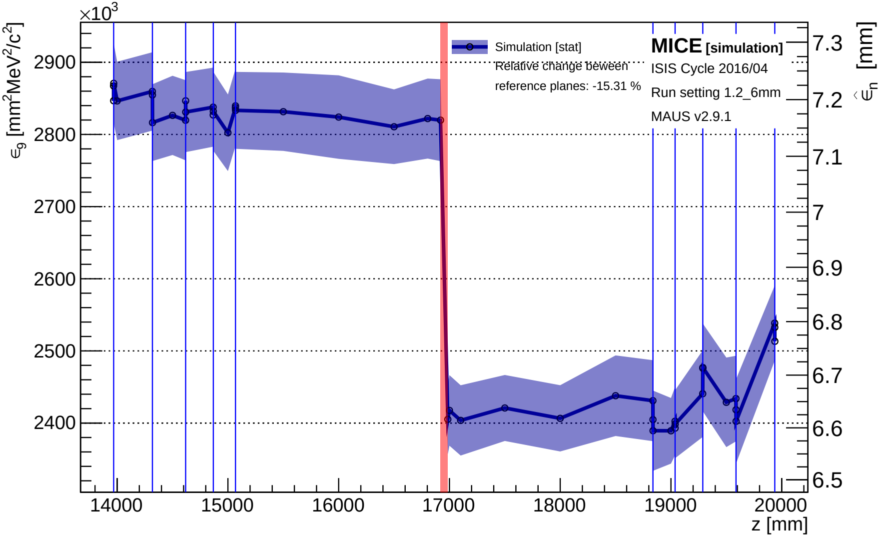




Density estimation at TKD station 5



Phase space volume evolution



Conclusions

Status of the amplitude-based analysis:

- Selecting the low amplitude core **gets rid of apparent emittance reduction** due to scraping and **apparent emittance growth** due to beam filamentation in the downstream section;
- A toy MC shows that the **exact same behaviour** is observed for the subemittance and fractional emittance as for the RMS definition;
- Method shows a **clean cooling signal** in a realistic MC.

Status of the non-parametric analysis:

- Systematic study well advanced, **kNN robust in 4D**, low error and bias for large samples with the rule-of-thumb k selection;
- Method applied to the toy MC to study its behaviour, **identical trend** as with the amplitude-based fractional emittance;
- Method also shows **cooling signal** in a realistic MC.

Back-up slides

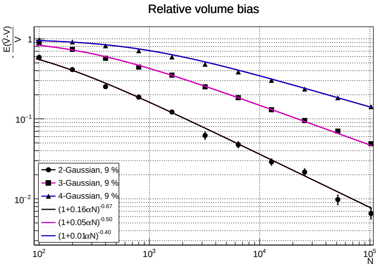
Convex hull volume uncertainty

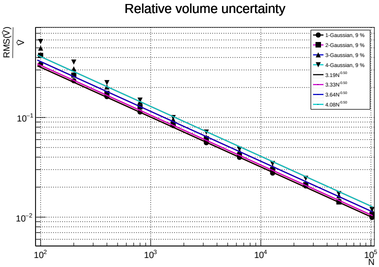
Computing the volume of the convex hull of a core fraction α of a set of N i.i.d. random d-Gaussian points yields

$$\frac{E[\hat{V}-V]}{V} \sim -(1+C^{-\frac{d+1}{2}}\alpha N)^{\frac{-2}{d+1}}, \qquad \boxed{\frac{\sigma_{\hat{V}}}{\hat{V}} \sim \sqrt{\frac{1-\alpha}{\alpha N}}\exp\left[Q_{\chi^2_d}(\alpha)/4\right]} \qquad (12)$$

The factor C is purely deterministic, albeit complex in nature...

$$C = \frac{d+1}{2(d+3)(d-1)!}\Gamma\left(\frac{d+3}{d+1}+d\right)\left(\frac{2\pi}{B\left(\frac{1}{2},\frac{d}{2}+1\right)}\right)^{\frac{2}{d+1}} \qquad (13)$$



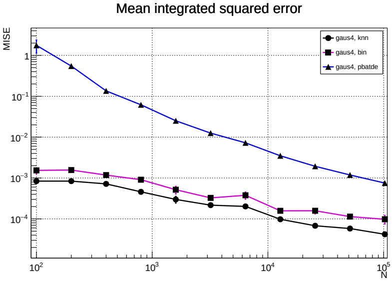


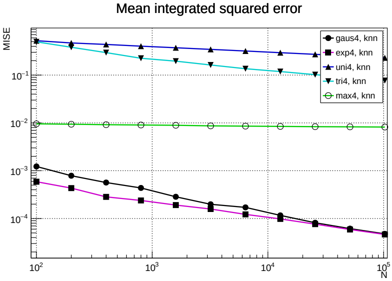
Density estimator Mean Integrated Squared Error

A critical characteristic of estimators is **consistency**. For large N, the estimator must converge to the true value that is estimated,

$$\lim_{N \to +\infty} \hat{\theta}_N = \theta. \tag{14}$$

Deviation from the true estimated distribution can be quantified by computing the Mean Integrated Squared Error (**MISE**).



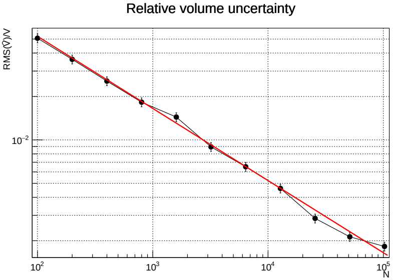


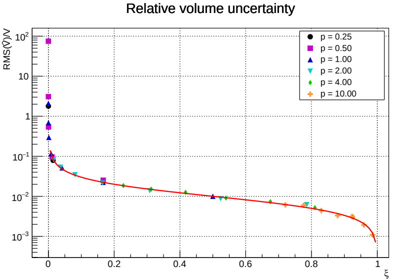
Contour volume reconstruction

In 4D, the most efficient way to estimate an arbitrary volume is to generate **random points** inside a 4-box that bounds the contour and to count the amount of points that are above the contour level.

This is a Binomial process which yields an uncertainty of

$$\boxed{\frac{\sigma_{\hat{V}}}{\hat{V}} = \sqrt{\frac{1-\xi}{\xi N}}}, \qquad \xi = \frac{N_{\text{in}}}{N}. \tag{15}$$





kNN contour volume bias and uncertainty

In addition to the counting error, which **can be tamed** by adding random points, there is a certain level of uncertainty related to the density estimation. The bias and variance satisfy

$$\frac{E[\hat{V} - V]}{V} \propto (\alpha N)^{-1/\lambda}, \qquad \frac{\sigma_{\hat{V}}}{\hat{V}} \sim \sqrt{\frac{1-\alpha}{\alpha N}\frac{\rho_0}{\rho_\alpha}}, \tag{16}$$

which are distribution dependent.