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A new Space-Charge Model for high-intensity Particle Beams

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14. September 2017

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Outline

1. Introduction
2. Basic Accelerator Physics
3. Space Charge Model
4. Beam Core Dynamics
5. Spectator Particle Dynamics
6. Conclusion

Introduction to High Power Accelerators

Introduction
Basic Accelerator Physics
Space Charge Model
Beam Core Dynamics
Spectator Particle Dynamics
Conclusion

European Spallation Source (Lund)

- neutron source
- High-intensity proton beam

Consequences of beam loss:

- heat load (cryogenics)
- radiation (maintanance)

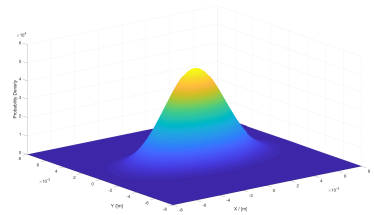
Life- and Material Sciences

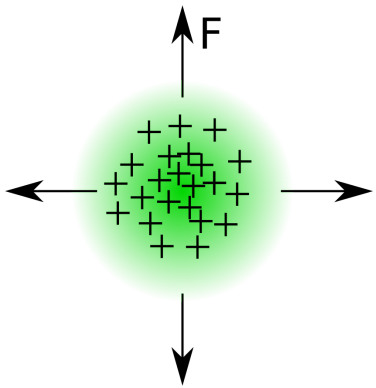
5 MW rms beam power loss threshold: 1 $\frac{W}{m}$

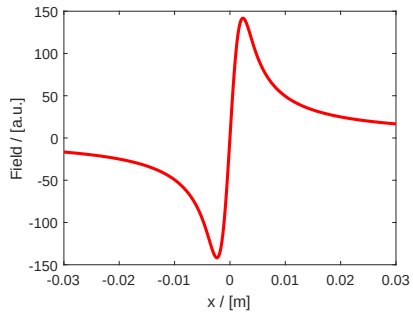
intensity-dependent effects

Space Charge

- same-charged particles $\Rightarrow$ Space Charge
- non-linear electric forces (e.g. Gauss)
- particles pushed to higher amplitudes $\Rightarrow$ halo
- halo is prone to be lost







Research Questions

What does the new space charge model address?

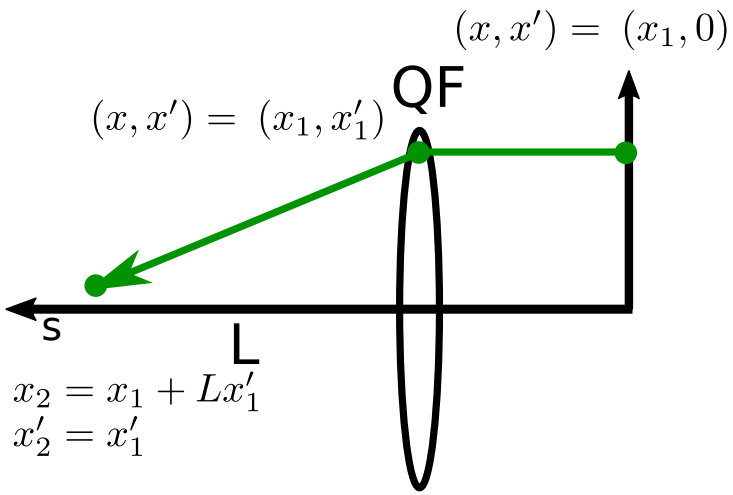
- Understand beam dynamics at high intensity
- Evolution of beam size
- Mechanics of beam halo formation
- Limits in beam current

The new model works generally, but now: Rings.

Propagation of Particles

Particle Transport

$$\begin{bmatrix} \bar{x} \\ \bar{x}' \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}$$



Linear optics (drift space, dipoles, quadrupoles)

R: lattice transfer matrix

$$R_{drift} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}$$

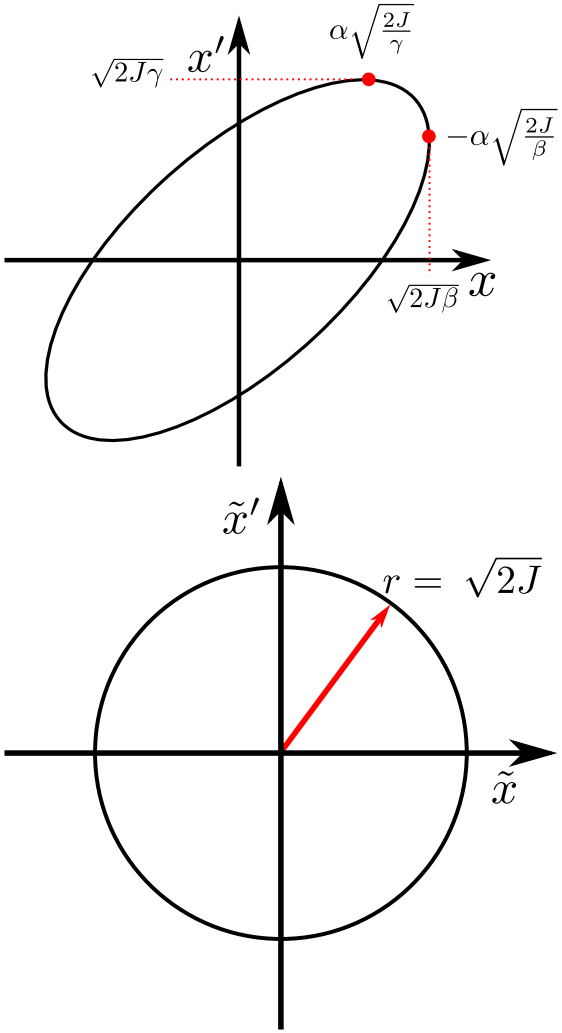
Distribution Transport

$$\bar{\sigma} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}^T;$$

$$\bar{\sigma} = R\sigma R^T$$

$$\sigma = \begin{bmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle xx' \rangle & \langle x'^2 \rangle \end{bmatrix}$$

Visualize Particle Oscillations



- phase space
- angle vs. position
- Particle trajectory in ring
- stable: follows ellipse

- normalized phase space
- particle action $J = \frac{\tilde{x}^2 + \tilde{x'}^2}{2}$
- has equivalent for distributions

Twiss Parameters
α, β, γ

β: *amplitude*
J: *action variable*

Distribution
$\epsilon = \langle J \rangle$; *emittance*
$\sigma_x = \sqrt{\epsilon\beta}$; *size*

Beam Focusing and Tune

- Quadrupole magnets focus the beam
- Tune: number of oscillations per turn
- equilibrium $\Rightarrow$ stable motion
- perturb equilibrium $\Rightarrow$ oscillations

Harmonic Oscillator Tune Q

for stability: fractional tune

Effects of space-charge:

- defocusing
- reduces tune

Space Charge Tune Shift

Existing Simulation Methods

1. Envelope equations

$$\frac{d^2 a}{ds} + a\kappa - \frac{F_{sc}}{a} - \frac{\epsilon^2}{a^3} = 0$$

2. Particle in Cell (PIC), multi-particle tracking $\Rightarrow$ computationally expensive

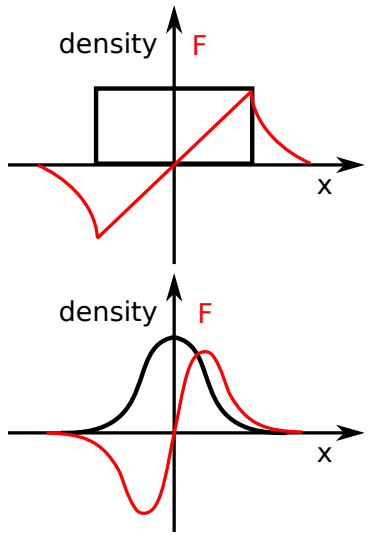
We complement these methods with a space-charge model that

- is valid for transverse, Gaussian beams
- includes non-linear space charge forces
- is fully analytic

Eq. of Motion

a(s): envelope amplitude

F_{sc}: space charge



Basis of the Model

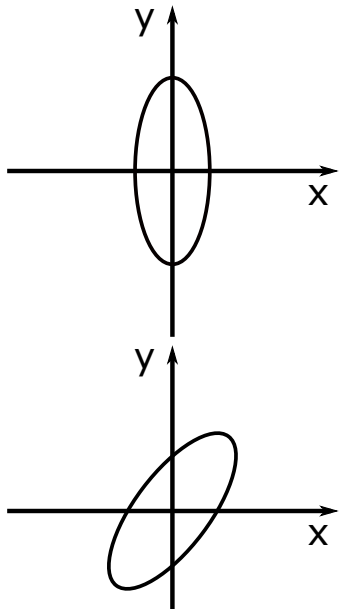
Electric field of a Gaussian particle distribution: (Bassetti & Erskine, 1980)

$$E_x - iE_y = -i\frac{Q}{2\epsilon\sqrt{2\pi\left(\sigma_x^2 - \sigma_y^2\right)}}\left[w\left(\frac{x+iy}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right) - e^{\left[-\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_x^2}\right]}w\left(\frac{x\frac{\sigma_y}{\sigma_x}+iy\frac{\sigma_x}{\sigma_y}}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right)\right]$$

- w: complex error function
- only valid for upright Gaussian beam

$\Rightarrow$ we use a covariant form $F_0 = f_3 + if_1$ to include tilt angles

$\Rightarrow$ cross-plane coupling



Self-Interaction of the Beam

1. particle distribution $\rightarrow \Psi(\vec{x}) = \frac{1}{(2\pi)^2\sqrt{\det\sigma}} e^{-\frac{1}{2}\sigma^{-1}_{mn}x_m x_n}$
2. electric field $\rightarrow F_0(x_1, x_3, \sigma) = f_3 + if_1$

$\Rightarrow$ beam matrix σ is identical for space charge

Particle experiences a kick:

$$\begin{aligned} \bar{x_1} &= x_1 \\ \bar{x_2} &= x_2 + f_1 \\ \bar{x_3} &= x_3 \\ \bar{x_4} &= x_4 + f_3 \end{aligned}$$

$$\left[x, x^{'}, y, y^{'}\right] = [x_1, x_2, x_3, x_4]$$

What happens to a beam distribution?

Space Charge Map

- Averaging over all particles

$$\sigma = \begin{bmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle xx' \rangle & \langle x'^2 \rangle \end{bmatrix}$$

$$\begin{aligned}
\bar{\sigma}_{11} &= \langle \bar{x}_1 \bar{x}_1 \rangle = \langle x_1 x_1 \rangle = \sigma_{11} \\
\bar{\sigma}_{12} &= \langle \bar{x}_1 \bar{x}_2 \rangle = \langle x_1 (x_2 + f_1) \rangle = \sigma_{12} + \langle x_1 f_1 \rangle \\
\bar{\sigma}_{13} &= \langle \bar{x}_1 \bar{x}_3 \rangle = \langle x_1 x_3 \rangle = \sigma_{13} \\
\bar{\sigma}_{14} &= \langle \bar{x}_1 \bar{x}_4 \rangle = \langle x_1 (x_4 + f_3) \rangle = \sigma_{14} + \langle x_1 f_3 \rangle \\
\bar{\sigma}_{22} &= \langle \bar{x}_2 \bar{x}_2 \rangle = \langle (x_2 + f_1)(x_2 + f_1) \rangle = \sigma_{22} + \langle 2 x_2 f_1 \rangle + \langle f_1^2 \rangle \\
\bar{\sigma}_{23} &= \langle \bar{x}_2 \bar{x}_3 \rangle = \langle x_3 (x_2 + f_1 \rangle = \sigma_{23} + \langle x_3 f_1 \rangle \\
\bar{\sigma}_{24} &= \langle \bar{x}_2 \bar{x}_4 \rangle = \langle (x_2 + f_1)(x_4 + f_3) \rangle = \sigma_{24} + \langle x_2 f_3 \rangle + \langle x_4 f_1 \rangle + \langle f_1 f_3 \rangle \\
\bar{\sigma}_{33} &= \langle \bar{x}_3 \bar{x}_3 \rangle = \langle x_3 x_3 \rangle = \sigma_{33} \\
\bar{\sigma}_{34} &= \langle \bar{x}_3 \bar{x}_4 \rangle = \langle x_3 (x_4 + f_3) \rangle = \sigma_{34} + \langle x_3 f_3 \rangle \\
\bar{\sigma}_{44} &= \langle \bar{x}_4 \bar{x}_4 \rangle = \langle (x_4 + f_3)(x_4 + f_3) \rangle = \sigma_{44} + \langle 2 x_4 f_3 \rangle + \langle f_3^2 \rangle
\end{aligned}$$

- linear terms $\langle x_k F_0 \rangle$
- quadratic terms $\langle f_1^2 \rangle$, $\langle f_3^2 \rangle$, $\langle f_1 f_3 \rangle$

$$\begin{aligned}
f_3^2 &= \Re\{F_0\}^2 = \left(\tfrac{1}{2}(F_0 + \bar{F}_0)\right)^2 \\
&= \tfrac{1}{4}\left(F_0^2 + \bar{F}_0^2 + 2 F_0 \bar{F}_0\right)
\end{aligned}$$

Linear Terms

$$\langle x_k F_0 \rangle = NK \int_{-\infty}^{\infty} d^4x \quad x_k \Psi(\vec{x}) F_0(x_1, x_3, \sigma);\ k = 1, 2, 3, 4$$

N,K: scaling

General Ansatz:

$$w(z) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} d\alpha\, e^{-\alpha^2 + 2i\alpha z}$$

Complex error fct integral form

$$\int_{-\infty}^{\infty} dx\, x^n e^{-\frac{x^2}{2\sigma^2}} = \left(\frac{\partial}{\partial B}\right)^n \int_{-\infty}^{\infty} dx\, e^{Bx - \frac{x^2}{2\sigma^2}} \Big|_{B=0}$$

completing squares parametric differentiation

General solution to linear terms:

$$\langle x_k F_0 \rangle = \frac{iNK}{2(\sigma_{11} - \sigma_{33} + 2i\sigma_{13})} \left[\sigma_{k1} + i\sigma_{k3} - \left(\frac{\sigma_{k1}(\sigma_{33} - i\sigma_{13}) + i\sigma_{k3}(\sigma_{11} + i\sigma_{13})}{\sqrt{\sigma_{11}\sigma_{33} - \sigma_{13}^2}} \right) \right]$$

Quadratic Terms

$$\langle F_0 \bar{F}_0 \rangle = (NK)^2 \int_{-\infty}^{\infty} d^4x \quad \Psi(\vec{x}) F_0 \bar{F}_0$$

$f_3^2 = \Re\{F_0\}^2 = \left(\frac{1}{2}(F_0 + \bar{F}_0)\right)^2 = \frac{1}{4}\left(F_0^2 + \bar{F}_0^2 + 2F_0\bar{F}_0\right)$

$\vdots$

fast-forward several weeks of work

$\vdots$

$w(z) = w(ax_1 + bx_3)$

$S(\sigma, a, b)$

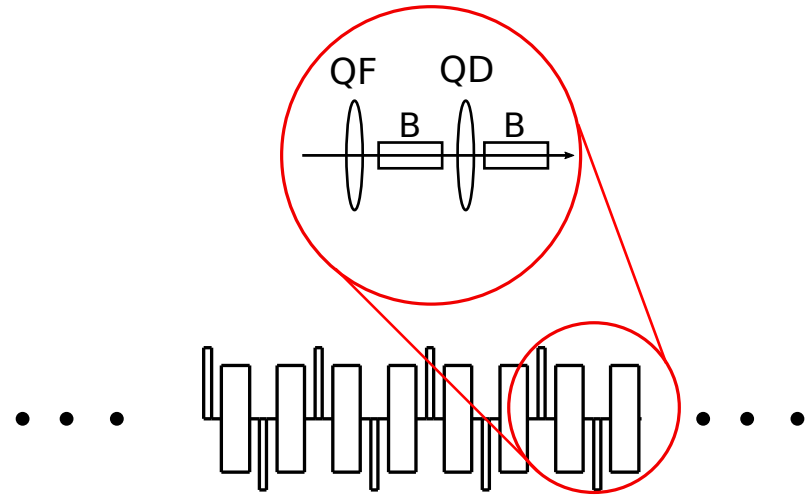
$$\langle F_0 \bar{F}_0 \rangle = \frac{(-NK)^2 \pi}{\sqrt{2(\sigma_{11} - \sigma_{33} + 2i\sigma_{13})}\sqrt{2(\sigma_{11} - \sigma_{33} - 2i\sigma_{13})}} \cdot \left[P(\sigma, 1, a, \bar{a}) - P(\sigma, 2, a, \bar{b}) - P(\sigma, 2, \bar{a}, b) + P(\sigma, 3, b, \bar{b})\right]$$

with

$$P(\sigma, n, a, b) = \frac{2}{n\pi} \frac{1}{\sqrt{S_{11}S_{22} - S_{12}^2}} \left[\frac{\pi}{2} - \arctan\left(\frac{S_{12}}{\sqrt{S_{11}S_{22} - S_{12}^2}}\right)\right]$$

Test Application: Simple Ring

- FODO-cell with 22.5° bending angle $\rightarrow$ 16 cells
- periodic structure



Turn-by-turn data:

- (equilibrium) beam sizes
- emittance growth
- space charge tune shifts

Beam motion with Space Charge

1. Slice machine lattice
2. Advance beam through a slice
3. Calculate space charge contribution T_{sc} in that slice
4. update beam matrix by adding the inhomogeneous term

R_s: sliced transfer matrix

Damping $D = I\ e^{-\frac{1}{\delta_{turn}}}$
Excitation E

Synchrotron radiation, quantized process, random

One slice:
$\sigma_{s,2} = R_{s,1}\sigma_{s,1}R_{s,1}^T + T_{sc,1}$

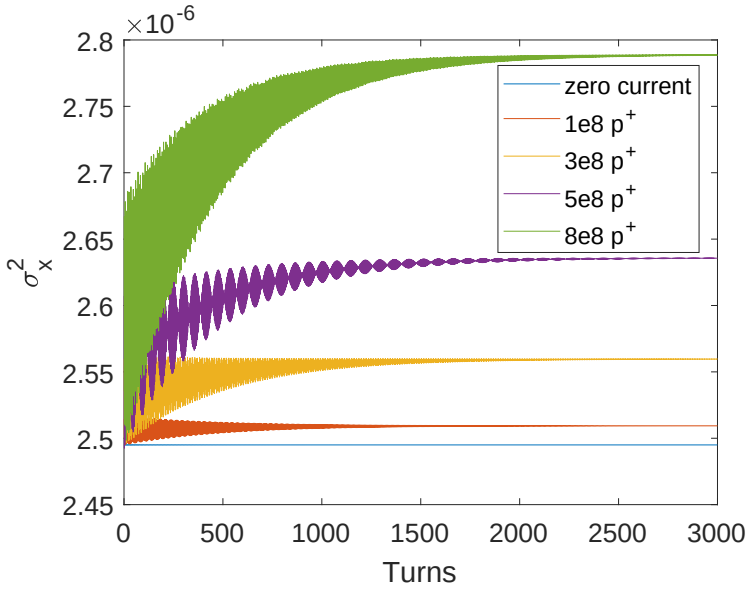
One turn:
$\sigma_{n+1} = \cdots R_{s,3}\left[R_{s,2}\left[R_{s,1}\sigma_n R_{s,1}^T + T_{sc,1}\right]R_{s,2}^T + T_{sc,2}\right]R_{s,3}^T + T_{sc,3}\cdots$

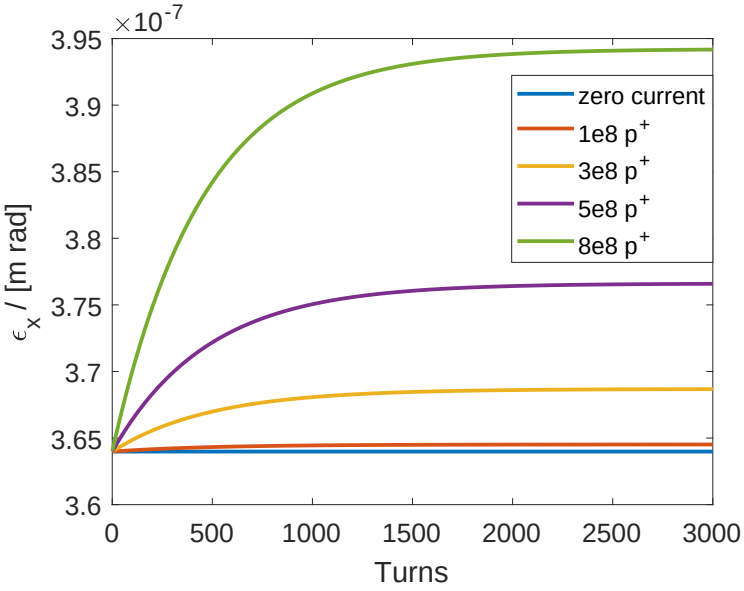
Effect of synchrotron radiation$\Rightarrow$ defines equilibrium

$\bar{\sigma}_{n+1} = D\sigma_{n+1}D^T + E \Rightarrow$ reach equilibrium

Turn-by-Turn Simulation

Turn-by-Turn data as function of number of particles in the beam during 'injection':

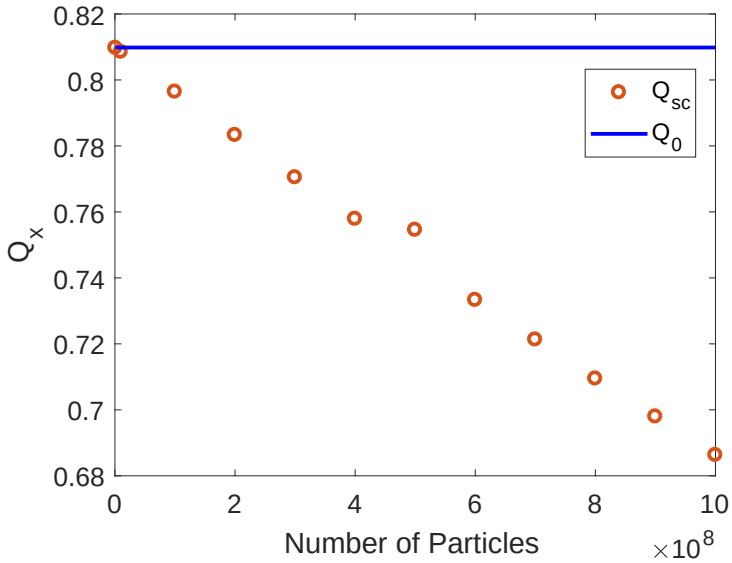


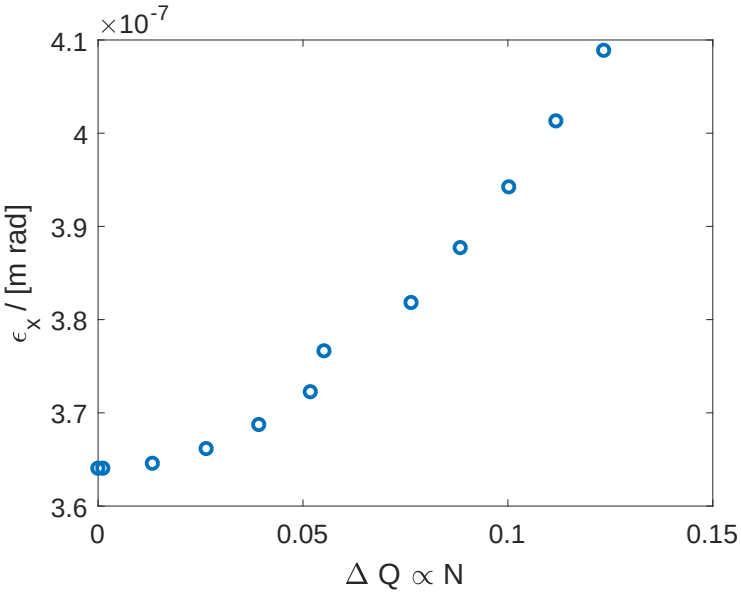


- rms beam size squared σ_x^2
- scale of effect $\approx$ 10 %

- beam emittance

Space Charge Tune Shift





for a round, uniform beam:

$$\Delta Q = -\frac{N r_0 L}{2\,(2\pi)^{\frac{3}{2}}\,\sigma_z \epsilon \beta^2 \gamma^3}$$

We obtain identical ΔQ by:

- making the beam as round as possible
- linearize the electric field

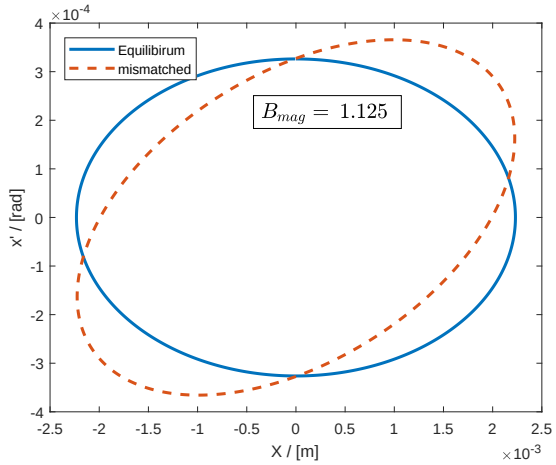
- emittance vs. tune shift $\Rightarrow$ current
- Quantifies space charge effect
- ϵ *vs.* ΔQ machine-'invariant'?

Spectator Particle Dynamics

How do the particles move to the tails of the Gaussian?

- Apply mismatch to equilibrium beam core
- Oscillating core with varying electric field
- Passive spectator particles ride on beam core

Matched: $B_{mag} = 1$
Mismatch B_{mag}

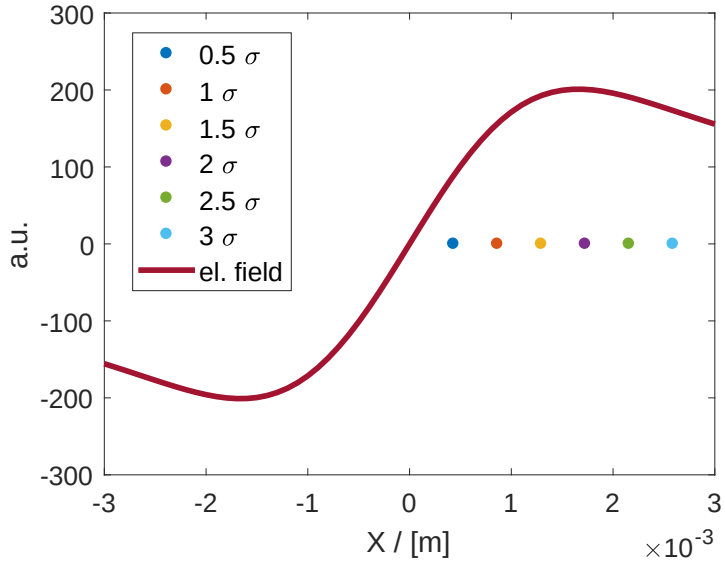


Method of investigation

- Poincaré-maps: dynamics of single particles
- parameters to vary:
  current, mismatch, coupling

Poincaré-Maps

- place single particles along position x
- observe particle coordinates on turn-by-turn basis
- dynamic behavior of single particles
- display in normalized phase space

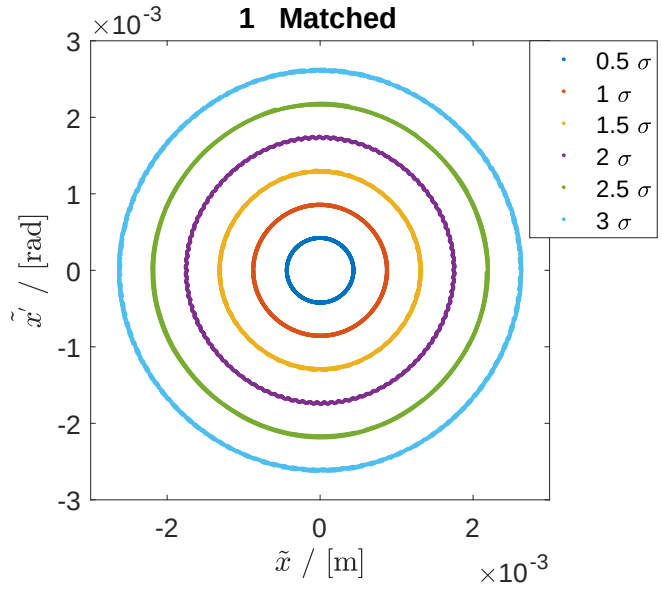


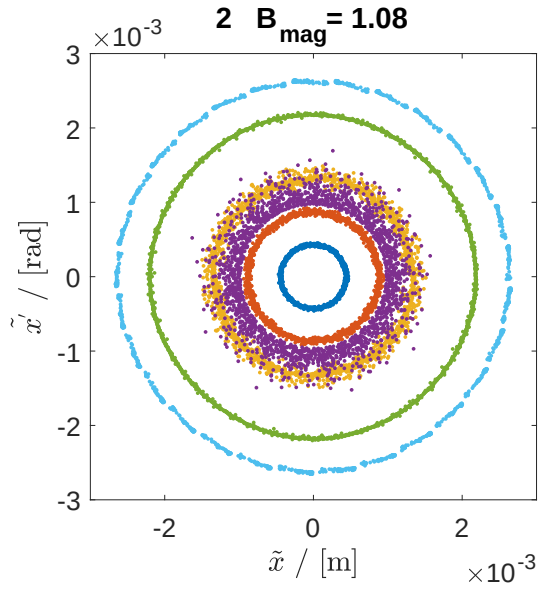
What kind of effects happen?

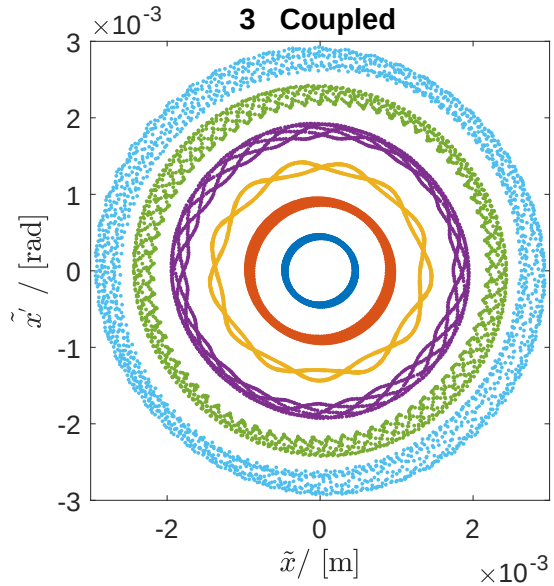
- amplitude-dependent tune shift
- resonance islands

unaffected particles draw circles

N= 4×10^8 protons per bunch





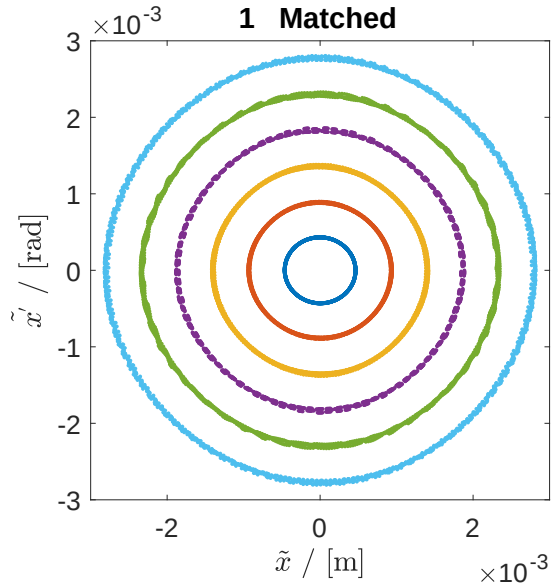


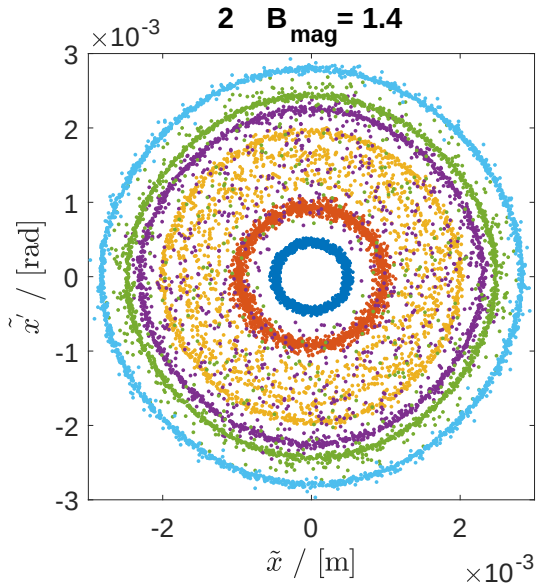
1
- almost perfect circles

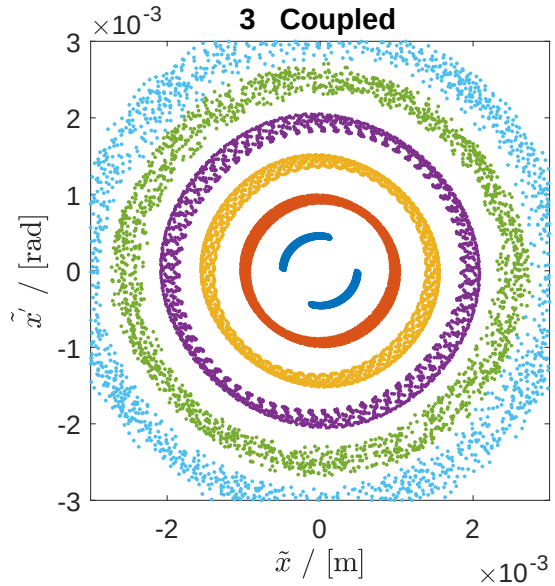
2
- disturbance between the 1σ- & 2σ-particle
- heavy displacement (to center)

3
- clear structures
- action exchange?

N= 10×10^8 protons per bunch







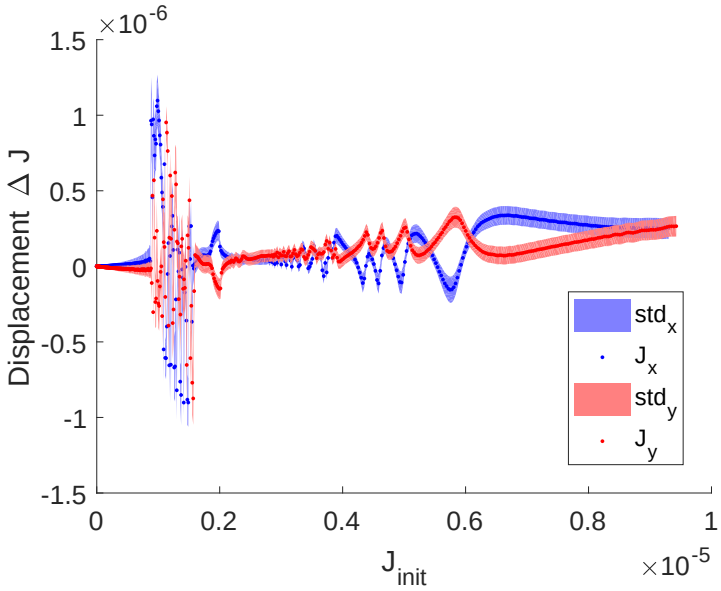
1
- smeared out circles
- bigger amplitudes

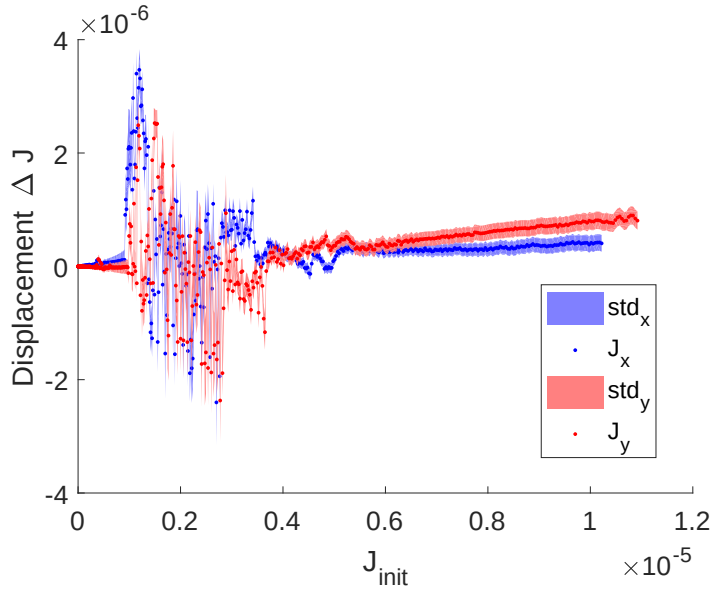
2
- heavy disturbance
- heavy displacement (to fringe)

3
- clear structures
- action exchange?

Analysis of Action Variables

- Particle Action $J = \frac{\tilde{x}^2 + \tilde{x'}^2}{2}$
- Plot $(\langle J_e \rangle - J_{init})$ vs. J_{init} for mismatched cases
- standard deviation: measure of disturbance





- N= 4×10^8 protons per bunch
- N= 10×10^8 protons per bunch

Summary

Introduction
Basic Accelerator Physics
Space Charge Model
Beam Core Dynamics
Spectator Particle Dynamics
Conclusion

1. Analytic space charge model for transverse Gaussian beams
2. Turn-by-turn beam core evolution in a test ring
3. Analysis of single particles, riding on beam cores

Beam Core Observations:

- Space charge tune shift of the core
- Machine performance (ϵ *vs.* ΔQ)

Single Particle Observations:

- Beam mismatch appears to be a major contributor to beam halo
- Beam current serves as 'lever arm'
- Analysis of particle actions quantifies chaos

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Future Investigations

- Include longitudinal plane
- Simulate planned machines, e.g. ESS accumulator ring (ESSnuSB)
- Effects of injection mismatch
- Magnet ripple (periodic disturbance, non-linear harmonic oscillator)
- stability and robustness of machines
- Linear accelerators

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